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*Full Length Research*

# Thermal properties of coulomb-type potentials in a magnetic field using the Phase – Integral model

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**ABSTRACT:** The work examined some Coulomb-type potentials under the influence of magnetic field potential and some select thermodynamic functions. The combination of the chosen Coulomb-type potentials and the magnetic field formed the generalized potentials which were put into the time independent radial Schrodinger equation and solution for the energy eigenvalues obtained using the Phase – Integral model. Four thermodynamic functions were considered and the results showed that an increase in the reduced parameters and magnetic field brings about a depreciation of the generalized potentials and the thermodynamic functions as well as the removal of degeneracy and integer decrease in the energy eigenvalues. The study also showed the change of the property of entropy in the classical domain as it enters the quantum mechanical system.

**Keywords**: Entropy, integer decrease, JWKB method, radial Schrodinger equation, reduced parameters, Thermodynamic functions.

## INTRODUCTION

Central potentials in quantum mechanics play a leading role in the prediction of physical fields, but no generalized potentials is capable of describing all quantum mechanical systems. As more potentials and their modifications develop to tackle existing problems, several other problems are also identified. Therefore, the formulation of potentials and several of their modifications is an ongoing practice and cannot be overemphasized. The use of the Coulomb–type potential was reported in the work of Ikot et al. (2014a). Maksimenko and Kuchin (2011) examined a combination of the harmonic oscillator, a linear potential and a Coulomb potential using the Nikiforov – Uvarov method to obtain the energy eigenvalues and wave function for large and small distances between particles in the bound state. A coulomb plus quadratic potential in a non – relativistic potential model was examined by Al-Oun *et al.* (2015) and characteristics of heavy quarkonia were discussed. Coulomb plus exponential type potential was examined by Yazarloo and Mehraban (2016) and results are in general, compactible with the literatures cited. An approximate analytical solution of the radial Schrodinger equation for the screened coulomb potential was obtained including the energy eigenvalues and the corresponding eigenfunctions by Budaca (2016). Edet *et al*. (2019) studied modified Kratzer potential plus screened Coulomb potential for the bound state solutions of the radial Schrodinger equation and some special cases were considered. Hassanabadi *et al*. (2014) published an exact solution of Klein–Gordon with the Posch Teller double–ring–shaped coulomb potential and suggested that the work can be applied in quantum chemistry and nuclear physics with requisite modifications. The study went further to state that the central potential failed to adequately study deformed nuclei and the molecular configuration of benzenes. Some of the studies have undergone several modifications and additions. Few are screened Kratzer potential (Ikhdair, 2009), Energy dependent molecular Kratzer potential (Ikot *et al*., 2020), Hellman potential and modified Kratzer potential (Edet *et al.,* 2019), Modified Kratzer molecular potential (Berkdemir *et al.,* 2006), Screened Kratzer potential (Amadi *et al*., 2020) and many more. Different methods and approach were used to investigate the different configurations of the potentials. Owing to the present of the centrifugal term or the nature of the effective potentials, the resulting Schrodinger equation, poses a challenge for closed form solutions. Though few exist for harmonic oscillators and hydrogen atoms, many approximate methods are abounded. The methods include but not limited to the point canonical transformation method (Abu-Shady, 2015), the Nikifarov-Uvarov method (De *et al*., 1992; Ikot *et al*., 2011), the numerical methods (Hassanabadi *et al*., 2017; Ixaru *et al.*, 2000), the asymptotic iteration method (Sandin *et al.,* 2016; Ciftci and Kisoglu, 2018), the super symmetry quantum mechanics (Ikhdair, 2011; Hamzavi *et al*., 2013), the factorization method (Ikot *et al.,* 2015; Sadeghi, 2007), the Hill determinant method (Okorie *et al*., 2018), the phase - integral or JWKB method (Chaudhuri and Mondal, 1995; Ghatak *et al.,* 1997; Ngiangia *et al*., 2018; Heading, 1962) and the series solution method (Ghatak *et al*., 1991; Kumar and Fakir, 2013). Since magnetic field is everywhere in the universe, its effect on other potentials are abounded. According to Ibekwe *et al*. (2020) and Ni and Chen (2002), the superconductivity state of an element or compound is reversed if it is introduced in the vicinity of a strong magnetic field strength. In like manner, it is suggested that a doped semiconductor material can returned to non-conductor by the influence of strong magnetic field. Filip (2015) observed that magnetic field alter the decay of resonance. Steiner and Ulrich (1989), in their study opined that the magnetic field influences the kinetics of the compound under consideration. Hayashi, (2004) stated that electron spin is affected by the presence of magnetic field. Ikhdair *et al*. (2015) examined molecular models under external magnetic field. Following the trend of investigations of the different types of potentials and their modifications, the authors intend to use the JWKB approximation method. The method started with the name WKB approximation method, named after the Wentzel, Kramers and Brillouin. However, in the book published by Froman and Froman (1965), they used the acronym JWKB. In the work of Froman and Fromam (1965), they also used the JWKB method in the validity for a triangular barrier. Ghatak and Lokanathan (2002) took a step further to explain that Jeffreys (1923), published an article entitled 'On certain approximate solutions of linear differential equations of second order and its connection formulae'. The application of JWKB solutions to problems in quantum mechanics came much later and was given by Wentzel (1929), Kramers (1926) and Brillouin (1926). Froman and Froman (1965) felt that since the solutions were first put forward by Wentzel (1929), it should be referred to as the JWKB method.

Thermodynamic properties are readily obtained from the partition function. The concept of the partition function is applicable in areas of chemical–physics challenges involving gases at different temperatures (Tiwari *et al*., 2012). Some thermodynamic functions such as entropy, specific heat, internal energy, and Helmholtz free energy have been studied using different potentials by authors. Some include Inyang *et al.* (2022), Khordad *et al.* (2022), Ikot *et al*. (2022), Ikot *et al*. (2018) and Inyang *et al.* (2021). Therefore, the aim of this study is to consider the thermodynamic functions on the proposed generalized potential in equation (7) which is an aggregate of some potential models of the type that contains the internuclear separation of the form $\frac{1}{r}$ , $\frac{1}{r^2}$ or both which fall into the category of the Coulomb–type potentials and in particular examine the effect of the reduced terms and the magnetic field on generalized potentials and the special cases. This the authors believed will add to existing literatures.

## THEORY OF COULOMB–TYPE POTENTIALS

According to Zettili (2001), the interaction between the electron and the proton as described by the Coulomb potential is stated as:

$$V(r) = -\frac{z_1 z_2 \alpha}{r} \quad (1)$$

Where $z_1$ and $z_2$ are projectile element and target element respectively, $\alpha$ is the coulomb constant and r represents the magnitude of the distance between the two particles.

The Coulomb potential has been used extensively in the study of hydrogen and its isotopes as well as many body problems. In the study of the radioactive decay of alpha particles, the potential as reported by Ghatak and Lokanathan (2002) is

$$V(r) = -\frac{2Z\alpha}{r} \quad (2)$$

Where Z is the atomic number of the daughter element.

It is reported that the Kratzer potential developed in 1934 which can be applied to gravitational and coulomb interactions, can be stated as (Hassanabadi et al. 2014)

$$V(r) = \frac{A}{r} + \frac{B_1}{r^2} \quad (3)$$

Where A and $B_1$ are potential parameters.

Following the Poschl – Teller double – ring – shaped coulomb potential as reported by Hassanabadi *et al*. (2014), it is stated as:

$$V(r) = -\frac{\beta}{r} + \frac{b}{r^2 Sin^2\theta} + \frac{A_1(A_1 - 1)}{r^2 Cos^2\theta} \quad (4)$$

Where $\beta$ , b and $A_1$ are potential parameters.

A thought is introduced, where the angular term is reduced to a rectangular or scalar term. The essence which may have chemical physics and nuclear physics prediction is also to examine the effect of such reduction or transformation with the main potential counterpart. The proposed potential reduction takes the form

$$V(r) = -\frac{\beta}{r} + \frac{b}{r^2 \lambda^2} + \frac{A_1(A_1 - 1)}{r^2 g^2} \quad (5)$$

Where $\lambda$ and g are reduced adjustable parameters to gauge the level of effect.

It is expected that if $A_1 = 1$, then the reduced Hartmann potential results

$$V(r) = -\frac{\beta}{r} + \frac{b}{r^2\lambda^2} \quad (6)$$

Having studied equations (1), (2), (3). (5) and (6), a modified generalized potential with the inclusion of magnetic field potential is stated as

$$V(r) = -\frac{qBm\hbar}{2\mu} - \frac{z_1 z_2 \alpha}{r} - \frac{2Z\alpha}{r} + \frac{A}{r} + \frac{B_1}{r^2} - \frac{\beta}{r} + \frac{b}{r^2\lambda^2} + \frac{A_1(A_1-1)}{r^2 g^2} \quad (7)$$

Where q is the charge of electron, B is the magnetic induction, m is the magnetic quantum number and $\hbar$ is the Planck's constant.

## Radial Schrödinger Equation with the effective potential

The potential is symmetric, therefore, the Schrodinger equation in spherical coordinate is suitable. However, the radial part of the Schrodinger equation in spherical coordinate is employed because the angular and azimuthal parts, play no role in the description of the potential considered. The work therefore, considered the radial Schrodinger equation of the form

$$\frac{1}{r^2} dr \left( r^2 \frac{dR(r)}{dr} \right) + \frac{2\mu}{\hbar} \left( E - V(r) - \frac{l(l+1)}{2\mu r^2} \right) = 0 \quad (8)$$

Where $l$ is the orbital quantum number and E is the energy eigenvalues of the system.

Equation (7) is put into equation (8) and the result is

$$\frac{1}{r^2}dr\left(r^2\frac{dR(r)}{dr}\right)+\frac{2\mu}{\hbar^2}\left(\begin{array}{l}E+\frac{qBm\hbar}{2\mu}+\frac{z_1z_2\alpha}{r}+\frac{2Z\alpha}{r}-\frac{A}{r}-\frac{B_1}{r^2}\\+\frac{\beta}{r}-\frac{b}{r^2\lambda^2}-\frac{A_1(A_1-1)}{r^2g^2}-\frac{l(l+1)}{2\mu r^2}\end{array}\right)=0 \qquad (9)$$

To apply the phase–integral or the JWKB method, a new radial function is defined as

$$U(r)=rR(r) \qquad (10)$$

Equation (10) is put into equation (9) and after simplification, reduced to

$$\frac{d^2U(r)}{dr^2}+k^2(r)U(r)=0 \qquad (11)$$

Where $k^2=\frac{2\mu}{\hbar^2}\left(E+\frac{qBm\hbar}{2\mu}+\frac{z_1z_2\alpha}{r}+\frac{2Z\alpha}{r}-\frac{A}{r}-\frac{B_1}{r^2}+\frac{\beta}{r}-\frac{b}{r^2\lambda^2}-\frac{A_1(A_1-1)}{r^2g^2}-\frac{l(l+1)}{2\mu r^2}\right)=0$

The quantization of the energy levels of the bound states for potential wells with no rigid walls (Zettili, 2001) is given by

$$(n+1)\pi=\int_{r_1}^{r_2}k(r)dr \qquad n = 1, 2. 3, …….. \qquad (12)$$

The value of k(r) is put into equation (12) and simplify, results into the form

$$\left(n+\frac{1}{2}\right)\pi\hbar=\sqrt{-(2E\mu+qBm\hbar)}\int_{r_1}^{r_2}\frac{\sqrt{-r^2+\beta_1r-\beta_2}}{r} \qquad (13)$$

Where $\beta_1=(A-\beta-\alpha-2Z\alpha)\left(\frac{2\mu}{2E\mu+qBm\hbar}\right)$

$$\beta_2=\left(-B_1-\frac{b}{\lambda^2}-\frac{A_1(A_1-1)}{g^2}-\frac{l(l+1)\hbar^2}{2\mu}\right)\left(\frac{2\mu}{2E\mu+qBm\hbar}\right)$$

If $r_1$ and $r_2$ are the roots of the polynomial $-r^2+\beta_1r-\beta_2$, then

$$-r^2+\beta_1r-\beta_2=(r-r_1)(r_2-r) \qquad (14)$$

Also, following the symmetric properties of roots of polynomial expression

$$-r^2+\beta_1r-\beta_2=-r^2+(r_1+r_2)r-r_1r_2 \qquad (15)$$

Then $\beta_1=r_1+r_2$ and $\beta_2=r_1r_2$ (16)

In the work of Griffiths (2005), a standard integral of the form stated below was used.

$$\int_{P_1}^{P_2}\frac{1}{r}\sqrt{(r-P_1)(P_2-r)}=\frac{\pi}{2}\left(\sqrt{P_2}-\sqrt{P_1}\right)^2 \qquad (17)$$

Using equation (17), equation (13) takes the form

$$2\left(n+\frac{1}{2}\right)\hbar=\sqrt{-\left(2E\mu+qBm\hbar\right)}\left(\sqrt{r_2}-\sqrt{r_1}\right)^2 \quad (18)$$

Similarly, using equation (15), equation (18) boils down to

$$2\left(n+\frac{1}{2}\right)\hbar=\sqrt{-\left(2E\mu+qBm\hbar\right)}\left(r_2+r_1-2\sqrt{r_1r_2}\right) \quad (19)$$

And finally, using equation (16), equation (19) is simplified to the form

$$2\left(n+\frac{1}{2}\right)\hbar=\sqrt{-\left(2E\mu+qBm\hbar\right)}\left(\beta_1-2\sqrt{\beta_2}\right) \quad (20)$$

Putting back the values of $\beta_1$ and $\beta_2$ into equation (20) and simplify, the energy eigenvalues takes the form

$$E_{n,l}=-\frac{qBm\hbar}{2\mu}+\frac{\mu\left(-A-\beta-z_1z_2\alpha-2Z\alpha\right)^2}{4\mu\left(B_1-\frac{b}{\lambda^2}-\frac{A_1\left(A_1-1\right)}{g^2}-\frac{l(l+1)\hbar^2}{2\mu}\right)-\left(n+\frac{1}{2}\right)^2\hbar^2} \quad (21)$$

**Energy eigenvalues of the special cases**

The special cases are in the absence of the other potentials under consideration**.**
**Coulomb potential;** adjustment of parameters, $(A=\beta=Z=B_1=b=0, A_1=1)$ reduced equation (21) to

$$E_{n,l}=-\frac{qBm\hbar}{2\mu}+\frac{\mu\left(-z_1z_2\alpha\right)^2}{\left(-2l(l+1)\hbar^2\right)-\left(n+\frac{1}{2}\right)^2\hbar^2} \quad (22)$$

In the absence of the magnetic field term, the expression is in agreement with the work of Zettili (2001)

**Alpha decay potential;** $(A=\beta=z_1z_2=B_1=b=0, A_1=1)$ , equation (21) reduced to

$$E_{n,l}=-\frac{qBm\hbar}{2\mu}+\frac{\mu\left(-2Z\alpha\right)^2}{\left(-2l(l+1)\hbar^2\right)-\left(n+\frac{1}{2}\right)^2\hbar^2} \quad (23)$$

The expression is the same with the work of Ngiangia et al. (2018)

**Reduced Hartmann potential;** $(A=z_1z_2=Z=B_1=b=0, A_1=1)$, equation (21) takes the form

$$E_{n,l}=-\frac{qBm\hbar}{2\mu}+\frac{\mu\left(-\beta\right)^2}{4\mu\left(-\frac{b}{\lambda^2}-\frac{l(l+1)\hbar^2}{2\mu}\right)-\left(n+\frac{1}{2}\right)^2\hbar^2} \quad (24)$$

**Reduced Poschl – Teller double – ring – shaped coulomb potential;** $(A=Z=z_1z_2=B_1=b=0,)$ , equation (21) results in

$$E_{n,l}=-\frac{qBm\hbar}{2\mu}+\frac{\mu(-\beta)^2}{4\mu\left(-\frac{b}{\lambda^2}-\frac{A_1(A_1-1)}{g^2}-\frac{l(l+1)\hbar^2}{2\mu}\right)-\left(n+\frac{1}{2}\right)^2\hbar^2} \tag{25}$$

**Kratzer potential;** $(\beta=Z=z_1z_2=b=0, A_1=1)$, equation (21) is expressed as

$$E_{n,l}=-\frac{qBm\hbar}{2\mu}+\frac{\mu(-A)^2}{4\mu\left(B-\frac{l(l+1)\hbar^2}{2\mu}\right)-\left(n+\frac{1}{2}\right)^2\hbar^2} \tag{26}$$

In the absence of the magnetic field term, the expression is consistent with the work of Ibekwe *et al*. (2020).

## Thermodynamic functions

To describe the thermodynamic properties of the system, the grand partition function of ensemble of particles determination is of necessity. The partition function is a statistical measure of the extent to which energy is distributed among the different states of a system or molecules and a function of the degeneracy of the system (Tiwari *et al.,* 2012). The partition function of an ensemble of a quantum mechanical system is given as

$$Z_{n,l}(T)=\sum_{n,l=0}^{\infty}Exp\left(-\frac{E_{n.l}}{k_BT}\right) \tag{27}$$

Where $k_B$ is the Boltzmann constant and T is the temperature of the system.

The thermodynamic functions whose properties to be considered for the description of the energy eigenvalues of the coulomb–type potential are; Helmholtz free energy F(T), internal energy U(T), entropy S(T) and the specific heat at constant volume $C_V(T)$.

The Helmholtz free energy is determined by the relation

$$F(T)=-\frac{1}{T}\ln Z_{n,l}(T)=\frac{E_{n,l}}{k_BT^2} \tag{28}$$

The internal energy is stated as

$$U(T)=-\frac{\partial}{\partial T}\ln Z_{n,l}(T)=-\frac{E_{n,l}}{k_BT^2} \tag{29}$$

The entropy is of the form

$$S(T)=-k_B\frac{\partial F(T)}{\partial T}=\frac{2E_{n,l}}{T^3} \tag{30}$$

The specific heat at constant volume results in

$$C_V(T)=k_B\frac{\partial U(T)}{\partial T}=-\frac{2E_{n,l}}{T^3} \tag{31}$$

## RESULTS AND DISCUSSION

Table 1 shows the relationship between the quantum numbers and the energy eigenvalues. As the quantum numbers increase, degeneracy is completely removed with integer decrease in the allowed energy values for the generalized potential. This is true because several studies are abounded in the removal of degeneracy of a system. If the effective potential is not zero, degeneracy may be removed completely (Ikot et al 2014b) or partially as the case of Zeeman effect.

Table 2 displays the values of the energy eigenvalues for the reduced Poschl – Teller – ring – shaped Coulomb potential. It reveals that degeneracy is not only removed but the energy eigenvalues decreased.

Table 3 is the results of the eigenvalues of the reduced Hartmann potential. The results reveal that integer decrease is observed in the values of the energy eigenvalues and degeneracy removed.

Figures 1 to 3 showed that as the reduced parameters $\lambda$ and g and the magnetic field potential B increased, the generalized potential depreciates which explained that the parameters are decaying potentials.

Figures 4 to 6 showed that an increase in the reduced parameters $\lambda$, g and B, result in the decrease of the reduced Poschl – Teller – ring – shaped Coulomb potential. The reason is that, the considered parameters offers resistance to the potential under consideration.

Figures 7 and 8 showed that an enhanced $\lambda$ and B parameters, depreciate the reduced Hartmann potential.

Figures 9 to 11 show an increase in the reduced parameters and the magnetic field could not break the energy barrier of the Helmholtz free energy under the given temperature condition. As a result, the Helmholtz free energy F(T) depreciate as the $\lambda$, g and B parameters are enhanced.

The thermodynamic internal energy U(T) decreased as the $\lambda$, g and B are increased as shown in Figures 12 to 14.

The entropy S(T) is reduced as the $\lambda$, g and B parameters are increased as reported in Figures 15 to 17. The graphs saturated abruptly because the considered parameters were able to decrease the entropy and by extension reduce the internal energy of the system. This result is in agreement with the work of Rogers (2009), where it was reported that the rate of yield of products in chemical reaction is enhanced in the presence of magnetic field. This result is in conflict with the additive property of entropy in classical systems.

The $\lambda$, g and B parameters increase as shown in Figures 18 to 20, reveal that the specific heat at constant volume $C_V(T)$ decreases and in turn reduce the internal energy.

For the reduced Poschl – Teller – ring – shaped Coulomb potential, an increase in $\lambda$, g and B parameters lead to a decrease in the thermodynamic functions of F(T), U(T), S(T) and $C_V(T)$ as shown in Figures 21 to 32. These observations showed that the thermodynamic functions are unsaturated and their energy barrier unstable.

Figures 33 to 40 displayed the effect of increase of the $g$ and $B$ parameters on the thermodynamic functions of F(T), U(T), S(T) and $C_V(T)$ within the framework of the reduced Hartmann potential. The results showed that only in Figure 38 that the entropy is correspondingly increases otherwise a decrease is observed in all. The result of Figure 38 is accidental because in quantum mechanical system, the additive property of entropy is violated.

The exclusion of the results and discussion of coulomb potential (Zettili 2001), alpha decay potential in magnetic field (Ngiangia *et al.,* 2018), Kratzer potential (Ibekwe *et al*. 2020) were deliberate. The reason is that, the mentioned potentials have been discussed in the cited literatures, though Zettili (2001) and Ibekwe (2020) did not consider magnetic field, Ngiangia (2018) did but they were added to show the dexterity of the Phase- Integral model which is in line with the objective of this study.

**Table 1.** Energy (J) values obtained from equation (21), $B = 0.5x10^{19}T$, $q = 1.602x10^{-19}C$, $\mu = \hbar = B_1 = A = b = \beta = 1$, $\lambda = 0.4$, $g = 0.6$ $\alpha = 2.30662x10^{-28}Nm^2$, $A_1 = 2$, $z_1z_2 = 4$, Z = 90.

| m | n | $l$ | $E_{n,l}$ |
|---|---|---|---|
| | 0 | 0 | -11.4561 |
| | 1 | 0 | -13.4561 |
| | 2 | 0 | -17.4561 |
| | | 1 | -18.4561 |
| | 3 | 0 | -23.4561 |
| | | 1 | -24.4561 |
| | | 2 | -26.4561 |
| 1 | | | |
| | 4 | 0 | -31.4561 |
| | | 1 | -32.4561 |
| | | 2 | -34.4561 |
| | | 3 | -37.4561 |
| | 5 | 0 | -41.4561 |
| | | 1 | -42.4561 |
| | | 2 | -44.4561 |
| | | 3 | -47.4561 |
| | | 4 | -51.4561 |

**Table 2.** Energy (J) values of Reduced Poschl – Teller double - ring – shaped coulomb potential obtained from equation (25), $B = 0.5x10^{19}T$, $q = 1.602x10^{-19}C$, $\mu = \hbar = b = \beta = 1$, $\lambda = 0.4$, $g = 0.6$, $A_1 = 2$.

| m | n | $l$ | $E_{n,l}$ |
|---|---|---|---|
| 1 | 0 | 0 | -3.41439 |
| | 1 | 0 | -3.91439 |
| | 2 | 0 | -4.91439 |
| | | 1 | -5.16439 |
| | 3 | 0 | -6.41439 |
| | | 1 | -6.66439 |
| | | 2 | -7.16439 |
| | 4 | 0 | -8.41439 |
| | | 1 | -8.66439 |
| | | 2 | -9.16439 |
| | | 3 | -9.91439 |
| | 5 | 0 | -10.9144 |
| | | 1 | -11.1644 |
| | | 2 | -11.6644 |
| | | 3 | -12.4144 |
| | | 4 | -13.4144 |

**Table 3.** Energy (J) values of Reduced Hartmann potential obtained from equation (24), $B = 0.5x10^{19}T$, $q = 1.602x10^{-19}C$, $\mu = \hbar = b = \beta = 1$, $\lambda = 0.4$.

| m | n | $l$ | $E_{n,l}$ |
|---|---|---|---|
| 1 | 0 | 0 | -2.0255 |
| | 1 | 0 | -2.5255 |
| | 2 | 0 | -3.5255 |
| | | 1 | -3.7755 |
| | 3 | 0 | -5.0255 |
| | | 1 | -5.2755 |
| | | 2 | -5.7755 |
| | 4 | 0 | -7.0255 |
| | | 1 | -7.2755 |
| | | 2 | -7.7755 |
| | | 3 | -8.5255 |

**Table 3.** Contd.

| | | |
|---|---|---|
| | **0** | **-9.5255** |
| | 1 | -9.7755 |
| **5** | 2 | -10.2755 |
| | 3 | -11.0255 |
| | 4 | -12.0255 |

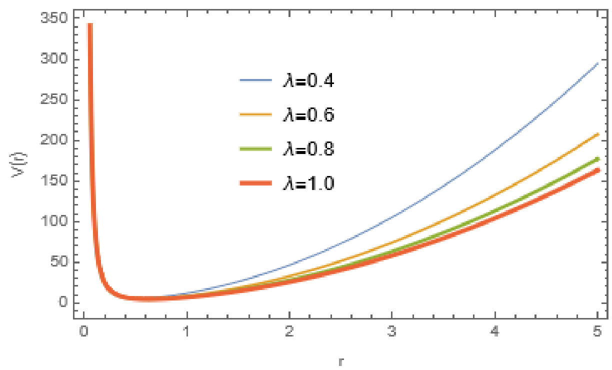


**Figure 1.** Plot of generalized potential as a function of the radius with $\lambda$ parameter varying.

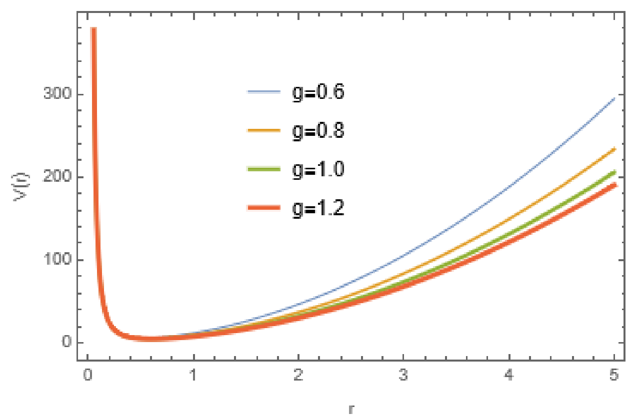


**Figure 2.** Plot of generalized potential as a function of the radius with g parameter varying.

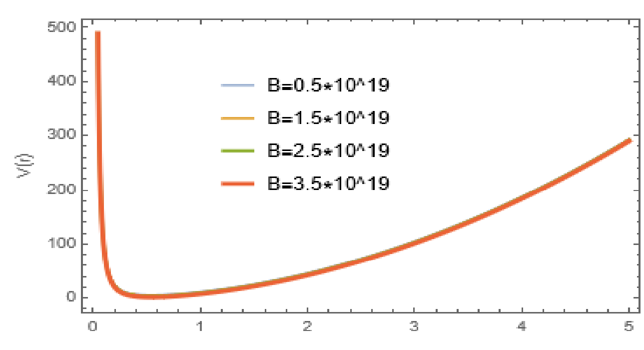


**Figure 3.** Plot of generalized potential as a function of the radius with magnetic field potential varying.

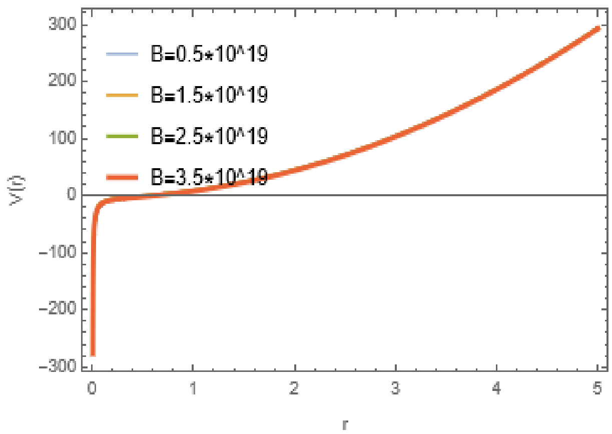


**Figure 4.** Plot of generalized potential $(\alpha = A = B_1 = 0)$ as a function of the radius with magnetic field varying.

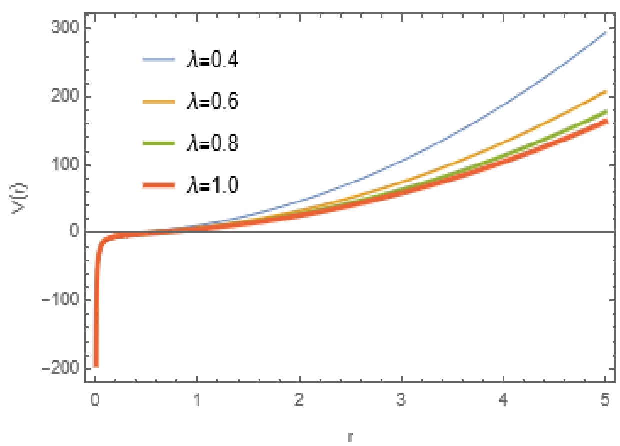


**Figure 5.** Plot of generalized potential $(\alpha = A = B_1 = 0)$ as a function of the radius with $\lambda$ parameter varying.

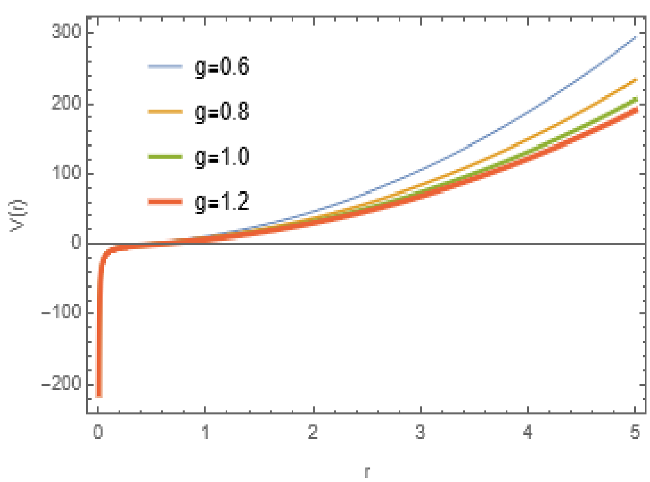


**Figure 6.** Plot of generalized potential $(\alpha = A = B_1 = 0)$ as a function of the radius with g parameter varying.

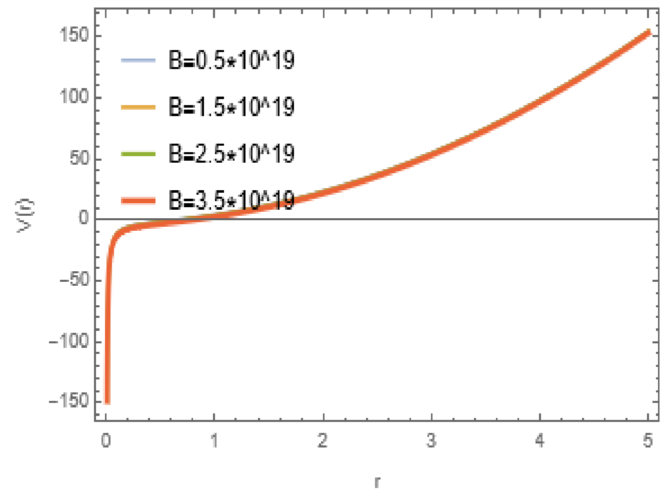


**Figure 7.** Plot of generalized potential $(\alpha = A = B_1 = A_1 = 0)$ as a function of the radius with magnetic field varying.

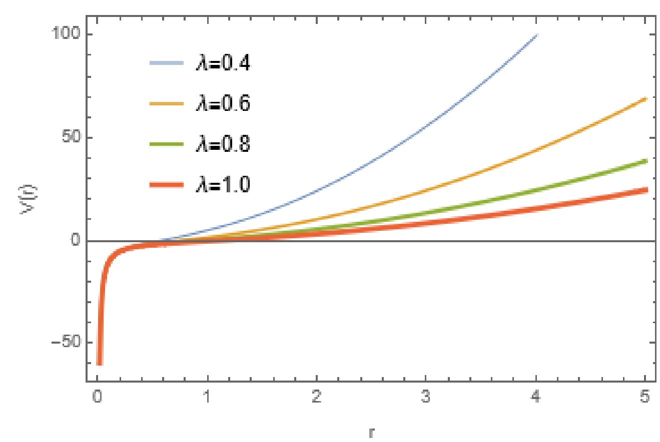


**Figure 8.** Plot of generalized potential $(\alpha = A = B_1 = A_1 = 0)$ as a function of the radius with $\lambda$ parameter varying.

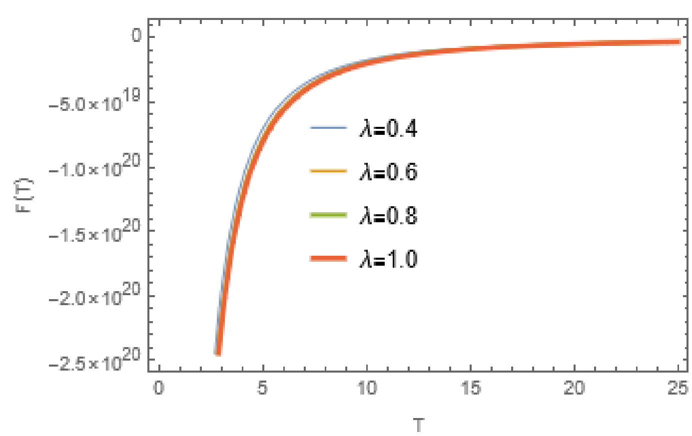


**Figure 9**. Plot of Helmholtz Free Energy as a function of temperature with $\lambda$ parameter varying.

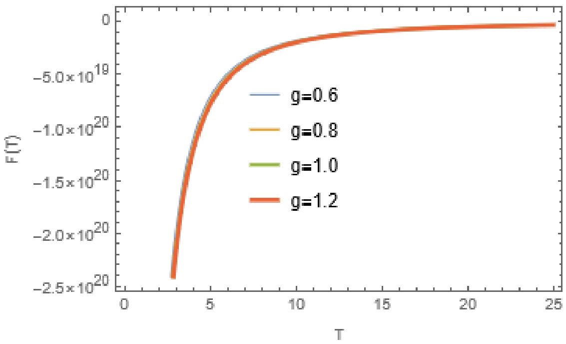


**Figure 10**. Plot of Helmholtz Free Energy as a function of temperature with g parameter varying.

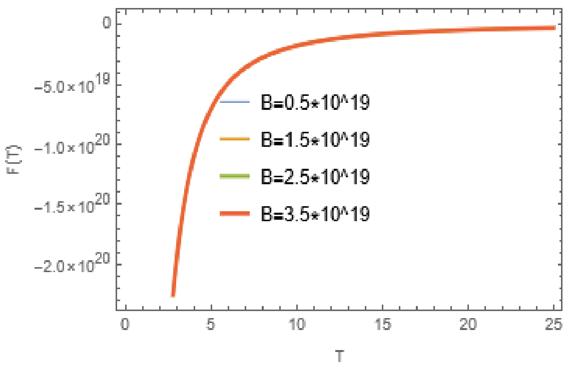


**Figure 11**. Plot of Helmholtz Free Energy as a function of temperature with B parameter varying.

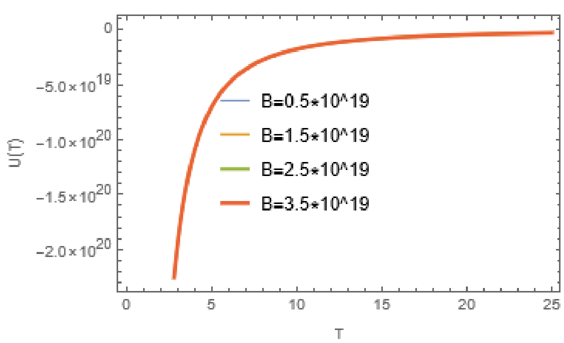


**Figure 12**. Plot of Internal Energy as a function of temperature with B parameter varying.

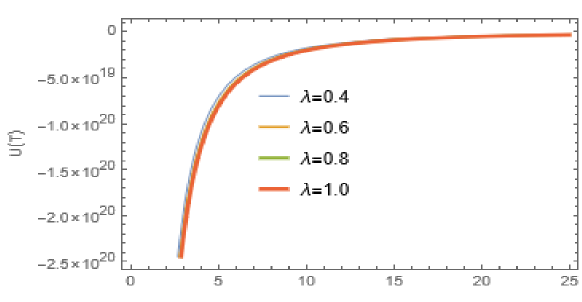


**Figure 13**. Plot of Internal Energy as a function of temperature with $\lambda$ parameter varying.

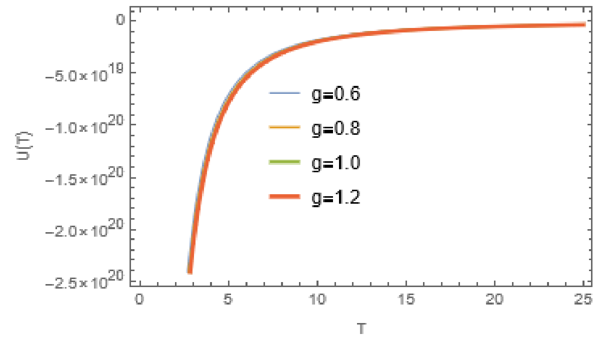


**Figure 14**. Plot of Internal Energy as a function of temperature with g parameter varying.

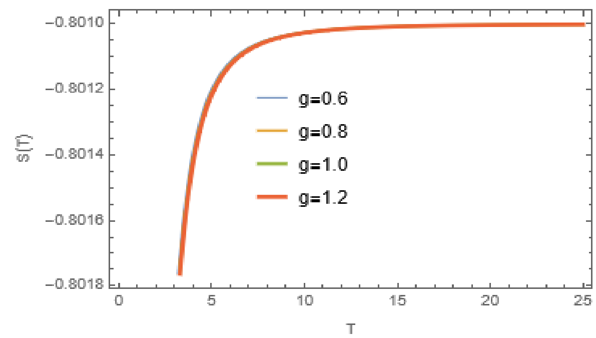


**Figure 15.** Plot of Entropy as a function of temperature with g parameter varying.

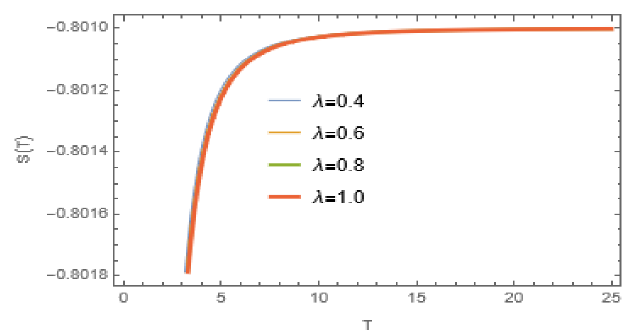


**Figure 16.** Plot of Entropy as a function of temperature with $\lambda$ parameter varying.

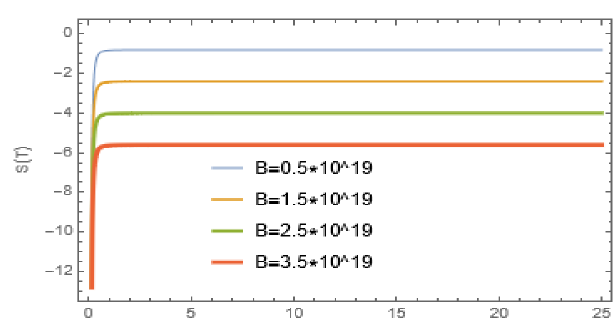


**Figure 17**. Plot of Entropy as a function of temperature with B parameter varying.

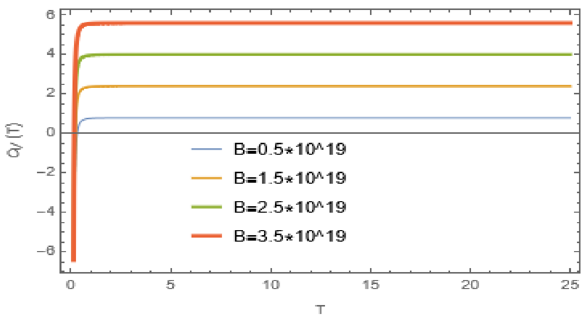


**Figure 18**. Plot of Specific heat at constant volume as a function of temperature with B parameter varying.

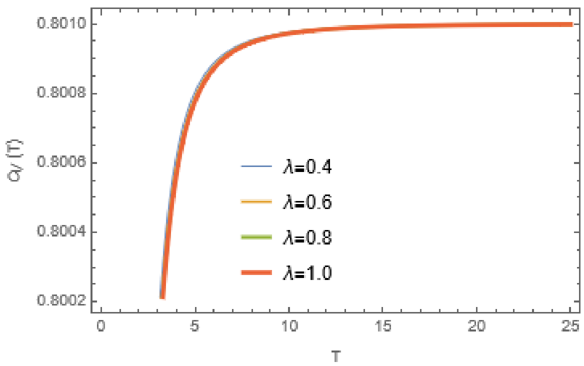


**Figure 19**. Plot of Specific heat at constant volume as a function of temperature with $\lambda$ parameter varying.

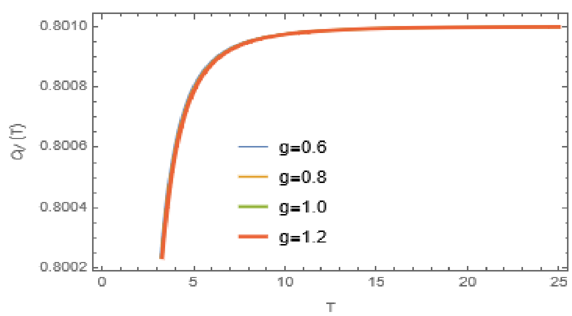


**Figure 20**. Plot of Specific heat at constant volume as a function of temperature with g parameter varying.

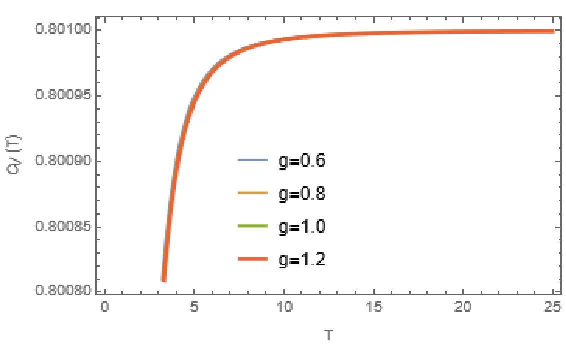


**Figure 21**. Plot of Specific heat at constant volume of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with g parameter varying.

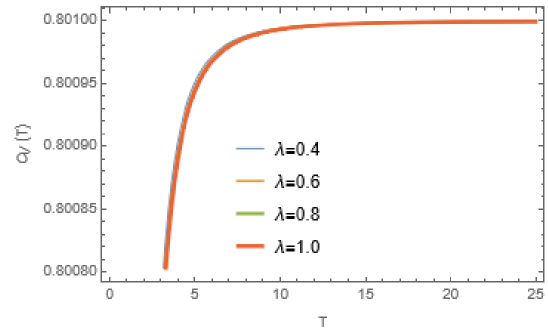


**Figure 22**. Plot of Specific heat at constant volume of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with $\lambda$ parameter varying.

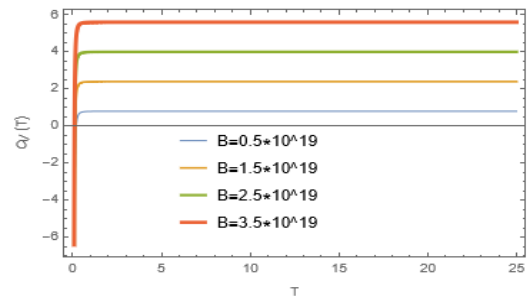


**Figure 23**. Plot of Specific heat at constant volume of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with B parameter varying.

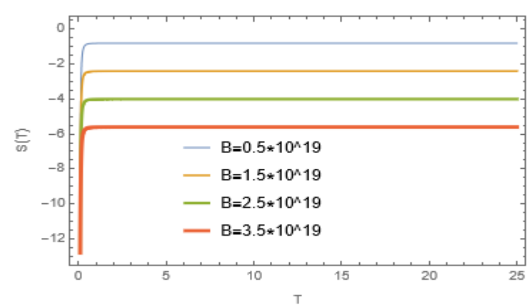


**Figure 24**. Plot of Entropy of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with B parameter varying.

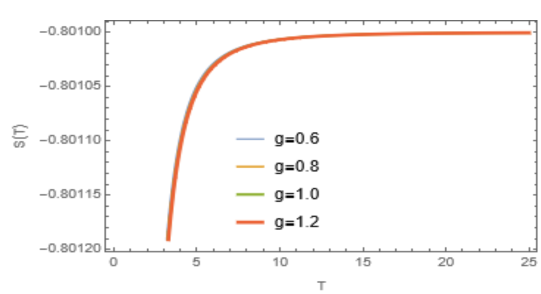


**Figure 25**. Plot of Entropy of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with g parameter varying.

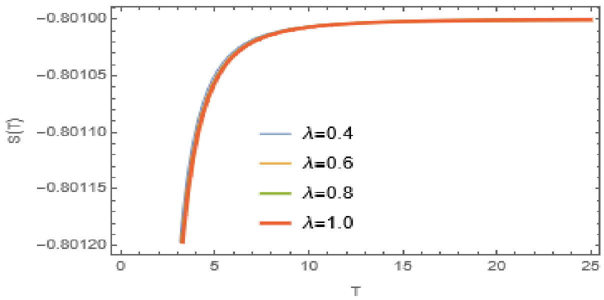


**Figure 26**. Plot of Entropy of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with $\lambda$ parameter varying.

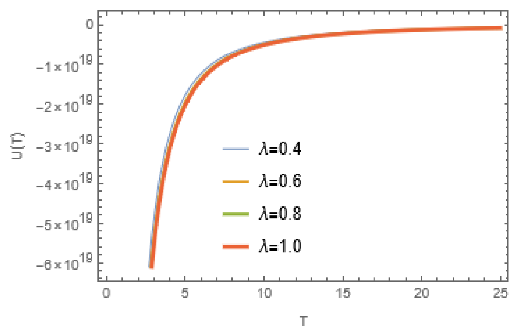


**Figure 27**. Plot of Internal energy of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with $\lambda$ parameter varying.

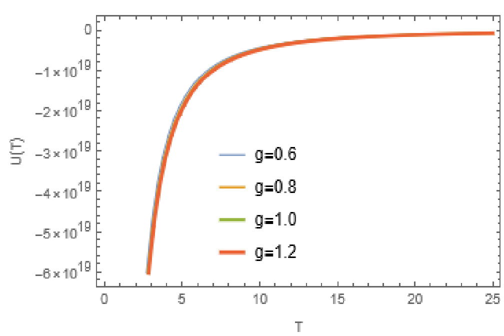


**Figure 28**. Plot of Internal energy of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with g parameter varying.

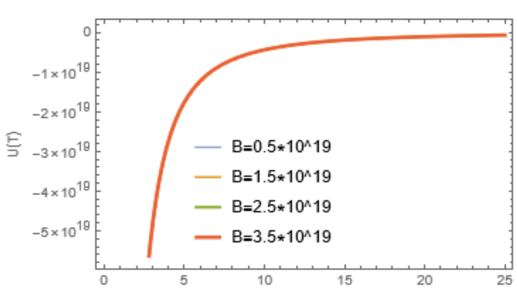


**Figure 29**. Plot of Internal energy of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with B parameter varying.

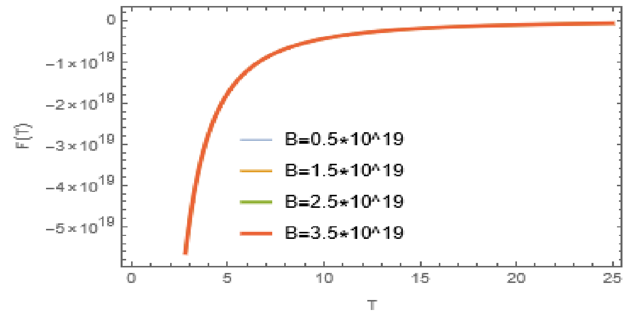


**Figure 30**. Plot of Helmholtz free energy of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with B parameter varying.

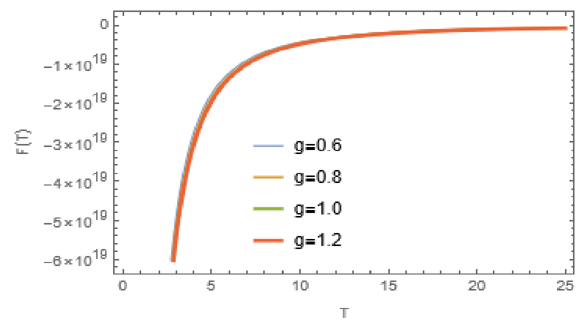


**Figure 31**. Plot of Helmholtz free energy of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with g parameter varying.

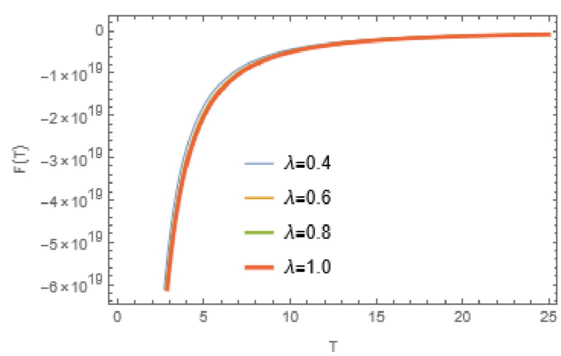


**Figure 32**. Plot of Helmholtz free energy of reduced Poschl – Teller double – ring – shaped Coulomb potential as a function of temperature with $\lambda$ parameter varying.

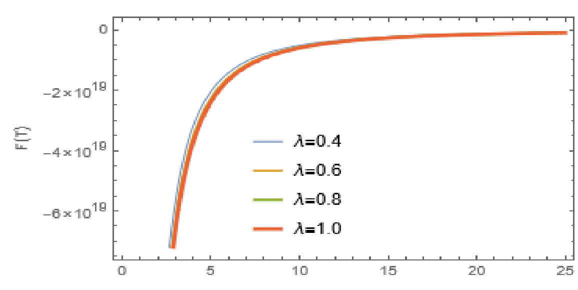


**Figure 33**. Plot of Helmholtz free energy of reduced Hartmann potential as a function of temperature with $\lambda$ parameter varying.

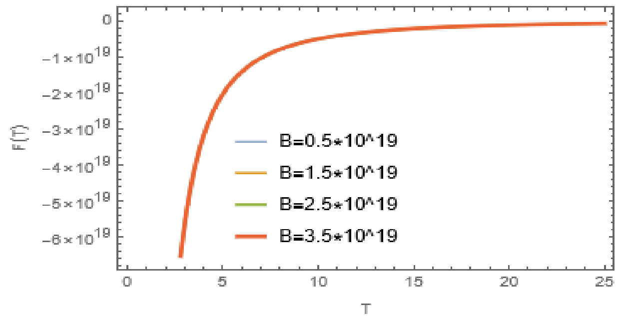


**Figure 34.** Plot of Helmholtz free energy of reduced Hartmann potential as a function of temperature with B parameter varying.

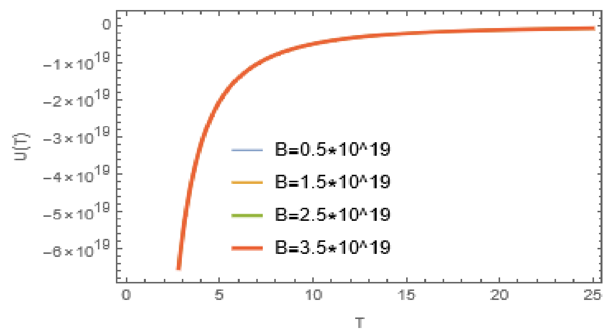


**Figure 35.** Plot of Internal energy of reduced Hartmann potential as a function of temperature with B parameter varying.

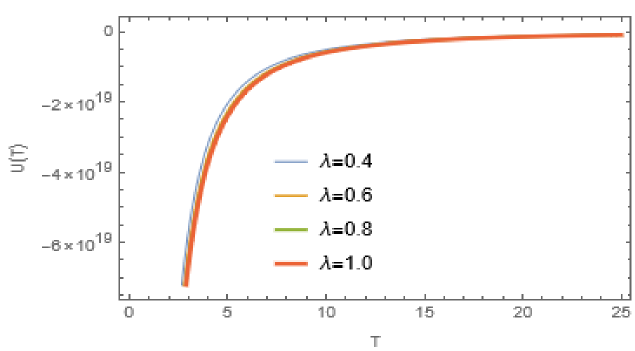


**Figure 36.** Plot of Internal energy of reduced Hartmann potential as a function of temperature with $\lambda$ parameter varying.

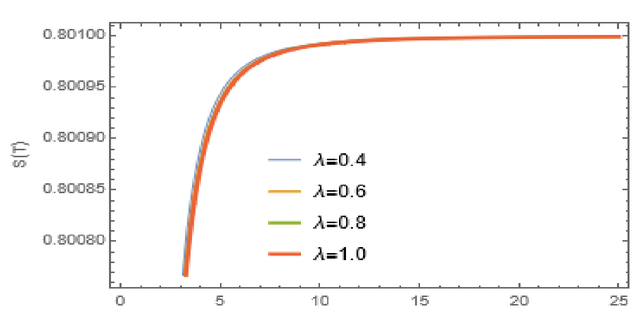


**Figure 37.** Plot of Entropy of reduced Hartmann potential as a function of temperature with $\lambda$ parameter varying.

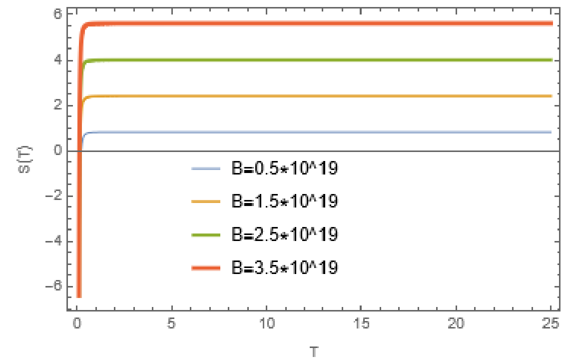


**Figure 38**. Plot of Entropy of reduced Hartmann potential as a function of temperature with B parameter varying.

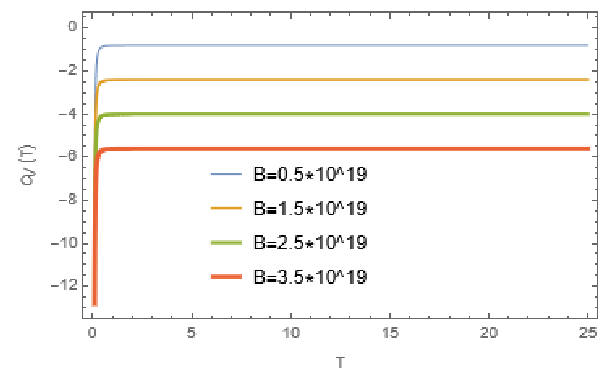


**Figure 39**. Plot of Specific heat at constant volume of reduced Hartmann potential as a function of temperature with B parameter varying.

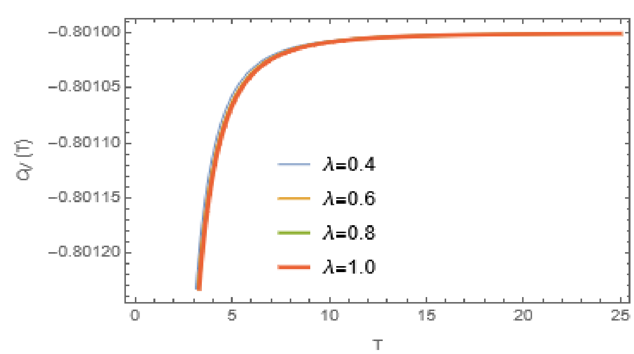


**Figure 40.** Plot of Specific heat at constant volume of reduced Hartmann potential as a function of temperature with $\lambda$ parameter varying.

## Conclusion

The kernel of this study was to consider the reduced parameters $(g,\lambda)$ and the magnetic field potential in the Coulomb – type potentials. The coinage of the title of the study is a combination of $\frac{1}{r}$ and $\frac{1}{r^2}$ type potentials. Generally, the presence of magnetic field potential in a given quantum mechanical system, destroyed or removed the degeneracy of the system which also laid credence to this study. The study also reveal that, the phase – Integral model can be used to tackle the determination of the eigenvalues and eigenfunctions of several combinations of the coulomb – type potentials in a single solution. The reduced parameters in the study, greatly affect not only the transformations but also added to existing works. Also, magnetic field is an indispensable factor in the study of classical and quantum mechanical systems. However, the JWKB method becomes handy in the description of Coulomb – like potentials as a whole owing to its simplicity

## CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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