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Full Length Research

A theoretical framework for nonlinear dynamics in finance systems: Implications for stability, control and synchronization

Cornelius Ogabi[1], Tijani Shehu[1], Babatunde Idowu[1], Rilwan Mustapha[2], Olasunkanmi Kesinro[1]* and Shu'aibu Muhammad[1]

[1]Department of physics, Lagos State University (LASU), Nigeria.
[2]Department of Mathematics, Lagos State University (LASU), Nigeria.

*Corresponding author. Email: olasunkanmi.kesinro@lasu.edu.ng

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ABSTRACT: It has been widely observed that most deterministic dynamical systems go into chaos for some values of their parameters. One of the most popular and widely used criteria is the conditional Lyapunov exponents, which constitute average measurements of expansion or shrinkage of small displacements along the synchronized trajectory. The Lyapunov characteristic exponents play a crucial role in the description of the behaviour of dynamical systems as they can be used to analyse the stability limit sets and to check sensitive dependence on initial conditions, that is, the presence of chaotic attractors. In this paper, Lyapunov stability theory and linear matrix inequalities (LMI) are employed to design control functions for the respective, control, and synchronization of the chaotic and hyperchaotic finance systems. The designed linear matrix inequalities (LMI) nonlinear controllers are capable of stabilizing the chaotic and hyperchaotic finance systems at any position as well as controlling it to track any trajectory that is a smooth function of time. The respective chaotic attractors were found to have a moderate value of the largest Lyapunov exponents (0.874959 s^{-1} and 0.650847 s^{-1}) with associated (Lyapunov) dimensions of 1.00 and 2.00 for the chaotic and hyperchaotic finance systems respectively. Based on Lyapunov stability theory and linear matrix inequalities (LMI), some necessary and sufficient criteria for stable synchronous behaviour are obtained and an exact analytic estimate of the threshold coupling, k_{th}, for complete chaos synchronization is derived. Finally, numerical simulation results are presented to validate the feasibility of the theoretical analysis.

Keywords: Chaos control, hyperchaos, linear matrix inequalities, Lyapunov exponents, Lyapunov stability, synchronization.

INTRODUCTION

Chaos refers to a type of complex dynamical behaviour that possesses extremely sensitive dependence to tiny variations of initial conditions, bounded trajectories in phase space and fractional topological dimensions (Rasappan and Vaidyanathan, 2014). Meanwhile,a hyperchaotic system is usually classified as a chaotic system with more than one positive Lyapunov exponent, indicating that the chaotic dynamics of the system are expanded in more than one direction giving rise to a more complex attractor (Li *et al*., 2005). There is an easy way (in principle) to check this fast sensitive dependence, namely by the calculation of Lyapunov exponents. The characteristic exponents give us an idea of whether a specific direction in the phase space is contracting or expanding. An expanding direction signifies a positive exponent and contracting a negative one. So, for that particular direction, the system goes through repeated stretching and folding processes. As a result of this, we cannot predict the long-term behaviour of the system given the initial conditions which is the very definition of chaos (Koshy-Chenthittayil, 2015). Owing to these properties, the accuracy to which the initial conditions are defined determines how accurately the future behaviour of a chaotic system can be predicted, even when the exact equations governing the system are known. Since computers can perform calculations only to a finite number of decimal places, the ability to predict the future behaviour of a chaotic system is always limited from a practical standpoint. This is an important reason that the accuracy of weather forecasts falls off dramatically as the forecast period increases, and it applies equally well to any physical or biological system that exhibits chaotic behaviour (Weiss *et al.,* 1993). Chaotic systems have been widely studied and applied in many real-world scenarios, such as weather prediction, industrial control, and market analysis (Luo and Song, 2016; Hua *et al*., 2019). In particular, they are often used in secure communication and encryption (Saberi-Nik *et al*., 2015; Wang *et al*., 2016; Wang *et al.,* 2020), as chaotic systems have some properties in unpredictability and initial state sensitivity (Lin *et al*., 2020).

In practice, however, it is often desired that chaos be avoided and/or that the system performance be improved or changed in some way. Given a chaotic attractor, one approach might be to make some large and possibly costly alteration in the system which completely changes its dynamics in such a way as to achieve the desired behavior (Ott *et al*., 1990). The ultimate boundedness of a chaotic system is very important for the study of the qualitative behavior of a chaotic system. In fact, except for the stability property, boundedness is also one of the foundational concepts of dynamical systems, which plays an important role in investigating the uniqueness of equilibrium, global asymptotic stability, the existence of the periodic solution, its control and synchronization, global exponential stability and so on (Saberi-Nik *et al*., 2015). But chaotic systems are not analytically solvable, studying them often relies on numerical methods. Many such methods have been devised to study the many facets of nonlinear systems (Datseris, 2018).

In the last decade, nonlinear dynamic analysis has developed in many disciplines such as economic sciences, ecology and environment, biology and engineering, etc. (Cao and Guo, 2020; Huang and Tan, 2021; Li *et al*., 2019; Qian and Hu, 2020). The pioneering research work by Pecora and Carroll (Pecora and Carroll, 1990), led to the field of chaos synchronization becoming an essential area, and since then, a lot of work has been done to achieve synchronization of various chaotic and hyperchaotic systems (Chen and Dong, 1998; Mkaouar and Boubaker, 2012; Pecora and Carroll, 1990; Shao *et al*., 2021). Recently, more outcomes on equilibrium-point stability, bifurcation, periodicity analysis and synchronization of various types of recurrent networks has been widely investigated in (Achouri *et al*., 2020; Aouiti *et al*., 2020; Mobayen *et al*., 2018). Consequently, numerous techniques for chaos stability and control as well as synchronization were proposed including observer-based control strategy, adaptive control, active control, backstepping design technique, variable structure control, linear matrix inequalities etc. (Ahmad and Shafiq, 2020; Anand and Sharma, 2022; Balootaki et al., 2020; Handa and Sharma, 2019; Khan and Nasreen, 2021; Kumar *et al*., 2021; Mkaouar and Boubaker, 2014; Mohadeszadeh and Delavari, 2017; Pai, 2019; Sharma *et al*., 2018; Shukla and Sharma, 2017; Xu *et al*., 2020; Zhao and Guo, 2015). The investigation of chaotic and hyperchaotic systems for example, Lorenz, Chen, Rossler, Chua, Sprott etc., has attracted much attention and achieved fruitful results due to their potential applications in scientific and engineering fields (Chaudhary and Sajid, 2022; Chen, 2002; Ge *et al*., 2003; Idowu *et al*., 2008; Lei *et al*., 2004; Chen and Li, 2004; Mofid *et al*., 2021; Vincent and Guo, 2012; Yamapi and Woafo, 2005). Ideally, chaotic and hyperchaotic financial systems include information on average profit margin, which simulate the actual complex and changeable financial market better and have a great impact on prominent economics at present. In the past years, the finance system has been found with rich phenomena (Chen, 2002; Van Dooren, 2003; Ge and Chen, 1996; Kocamaz *et al*., 2015) and, it exhibits a variety of interesting dynamical behaviours that span the range from regular to chaotic motions (Boukabou, 2008; Chen, 2002; Filali *et al*., 2012; Ge and Chen, 1996; Lopez-Mancilla and Cruz-Hernandez, 2005).

This paper intends to develop a theoretical framework to study the behavior of nonlinear dynamics in finance and the deductions of stability and control of chaos and hyperchaos. Since chaos/hyperchaos control is concerned with the use of some designed control input to modify the characteristics of a parameterized nonlinear system so that the system becomes stable at a chosen position or tracks a desired trajectory. From the viewpoint of control, synchronization of chaotic and hyperchaotic systems is somewhat a great task due to their nonlinear behavior and sensitivity to the initial values (Sun *et al*., 2018). Therefore, chaos and hyperchaos synchronization approaches based on linear state feedback control laws are applied due to their simple implementation. A set of algebraic synchronization conditions are derived and solved using the Lyapunov approach with suitable linear matrix inequalities (LMI) (Karami *et al*., 2021; Mkaouar and Boubaker, 2014). In Ref. (Olusola *et al*., 2010), a linear state error feedback approach based on Lyapunov stability theory and linear matrix inequality (LMI) is proposed (Liao and Wang, 2007), to analyze the stability of the synchronized state and also determine sufficient criteria for stable synchronous behaviour in finance systems. This method is used because it is known that many engineering optimization problems can be easily translated into linear matrix inequality (LMI); a wide variety of problems arising in system and control theory can be reduced to a few standard convex or quasi-convex optimization problems involving LMI. The resulting optimization problem can be solved numerically with very high efficiency (Boyd *et al*., 1994). Moreover, the Lyapunov methods which are traditionally applied to the analysis of system stability can just as well be used to determine threshold coupling, k_{th}, at which global synchronization could be reached in master-slave or mutually coupled oscillators. Critical coupling for the on-set of stable synchronization in coupled or driven oscillators is relevant for various scientific and technological applications (Stefanski, 2009).

In this paper, the synchronization of unidirectionally coupled finance systems was considered. A novel stability criterion using Lyapunov stability theory and linear matrix inequality (LMI) was proposed to determine the threshold coupling, k_{th}, at which full and stable synchronous behaviour could be reached in the master-slave coupled chaotic and hyper-chaotic finance systems. The advantage of this method is that the coupling parameters of the system can be obtained at the same time by solving the LMI without predetermining them to check the criterion. Furthermore, the LMI can be easily solved by various optimization algorithms. Sufficient criteria can be applied to directly design the coupling strength resulting in the synchronization.

The rest of the paper is structured as follows: in the next section, the synchronization scheme is presented, while section 3 is devoted to synchronization threshold and stability criteria, section 4 is devoted to numerical results and discussions and the paper is concluded in section 5.

MODEL AND SYNCHRONIZATION PRELIMINARIES

Chaotic finance system

Here, we consider the 3D chaotic finance system described as in the following equations:

$$\begin{aligned} \dot{x}_1 &= x_3 + (x_2 - a)x_1 \\ \dot{x}_2 &= 1 - bx_2 - {x_1}^2 \\ \dot{x}_3 &= -x_1 - cx_3 \end{aligned} \tag{1}$$

Where $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ are state space variables and a, b, c are positive real constants they represent the interest rate, investment demand, price exponent, per investment cost and elasticity of demands of commercials respectively. The nonlinear finance given by Eq. (1) exhibits varieties of dynamical behavior including chaotic motion for the following parameter values a = 0.8, b = 0.2 and c = 1.9.

To facilitate the present analysis, we express system (1) in the following vector form:

$$\dot{x} = Ax - f(x) + G(x) \tag{2}$$

In order to examine the synchronization between two unidirectional coupled finance systems, a master-slave synchronization scheme was constructed for two identical chaotic finance by linear state error feedback controllers in the following form:

$$\begin{aligned} M &: \dot{x} = Ax - f(x) + G(x) \\ S &: \dot{y} = Ay - f(y) + G(y) + u(t) \\ C &: u(t) = K(x - y), \end{aligned} \tag{3}$$

Where $u = K(x - y)$ is the linear state feedback control input and $K \in \mathbb{R}^{3\times3}$ is a constant control matrix that determines the strength of the feedback into the response system. By defining the synchronization error variable as the difference between the relevant dynamical variables given by Eq. (4):

$$e = x - y \tag{4}$$

we obtain the error dynamics for the master-slave system (3) as:

$$\dot{e} = \big(A - K + M(x, y)\big)e \tag{5}$$

Where $M(x,y) = \begin{pmatrix} x_1x_2 - y_1y_2 \\ -({x_1}^2 - {y_1}^2) \\ 0 \end{pmatrix}$ and $G(x,y) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

The chaotic finance system (1) has three equilibrium points:

$$f_1 = \left(0, \frac{1}{b}, 0\right), \; f_{2,3} = \left(\pm\sqrt{\left(\frac{c-abc-b}{c}\right)}, \left(\frac{1+ac}{c}\right), \mp\frac{1}{c}\sqrt{\left(\frac{c-abc-b}{c}\right)}\right)$$

Thus, the Jaccobian matrix of the chaotic system is given by

$$J = \begin{vmatrix} x_2 - a & x_1 & -1 \\ -2x_1 & -b & 0 \\ 1 & 0 & -c \end{vmatrix} \qquad (6)$$

At the fixed points, the respective eigenvalues are given by $\lambda_{1n} = -b,\ \lambda_{2n,3n} = \frac{(1/b - a - c) \pm \sqrt{(1/b - a + c)^2 - 4}}{2}$. Then, for a = 0.8, b = 0.2 and c = 1.9, the eigenvalues are estimated as follows: $(\lambda_{11} = -0.2000, \quad \lambda_{21} = -1.7314,\ \lambda_{31} = 4.0314)$, $(\lambda_{21} = -1.5848, \quad \lambda_{22} = 0.0055 + 1.3273i,\ \lambda_{23} = 0.0055 - 1.3273i)$ and $(\lambda_{31} = -1.5848, \quad \lambda_{32} = 0.0055 + 1.3273i,\ \lambda_{33} = 0.0055 - 1.3273i)$ respectively.

In the absence of the control matrix, K, Eq. (5) would have three equilibrium points at $(0{,}5{,}0)$, $(\pm 0.8572, 1.3263, \mp 0.4511)$. The aim of this study is to choose the appropriate coupling matrix, K, such that the trajectories of the master system, $x(t)$, and slave one, $y(t)$, satisfy

$$\lim_{t\to\infty} \parallel e \parallel = \lim_{t\to\infty} \parallel x(t) - y(t) \parallel = 0 \qquad (7)$$

Where || $*$ || represents Euclidean norm of a vector.

Procedure for calculation of Lyapunov exponents

Now to obtain the trajectories of points on the surface of the sphere, we consider the action of the linearized system on points very close to the fiducial trajectory. In fact, the principal axes are defined by the evolution via the linearized equations of an initially orthonormal vector frame anchored to the fiducial trajectory (Wolf *et al*., 1985). The formal way to describe how these perturbations react is with the use of partial derivatives. To set up the variational equations we would need to describe the variations. For this, consider the following matrix:

$$[\delta] = \begin{bmatrix} \delta_{x1} & \delta_{y1} & \delta_{z1} \\ \delta_{x2} & \delta_{y2} & \delta_{z2} \\ \delta_{x3} & \delta_{y3} & \delta_{z3} \end{bmatrix} \qquad (8a)$$

Where δ_{xn} is the component of the x variation that comes from the n^{th} equation. The column sums of this matrix are the lengths of the x, y and z coordinates of the evolved variation. The rows are the coordinates of the vectors into which the original x, y and z components of the variation have evolved. Therefore, the linearized equation for the variation in the finance chaotic 3D and 4D systems is given by:

$$\dot{\delta}_{xn} = \partial A \delta_{xn} \qquad (8b)$$

Where ∂A is the respective Jaccobian matrices for 3D and 4D systems. In addition to the original system of n nonlinear equations we will have an additional n^2 linearized equations. The system now has $n + n^2$ = $(n$ + 1) equations. To implement the procedure mentioned initially for creating the fiducial trajectory we solve the new system of $n(n$ + 1) differential equations with any numerical ode algorithm, e.g., Runge-Kutta 4, for some initial conditions and a time range $t_{start} + t_{start}.h$. Where $tstart$ denotes the initial time and h denotes the time step (Wolf *et al*., 1985).

Threshold and criteria for synchronization

Here, we have employed the Lyapunov's direct method and linear matrix inequality (LMI) (Horn and Johnson, 1991) to establish some criteria for global chaos synchronization in the sense of error system (5). The classical method of Lyapunov stability theory which employs Lyapunov functionals was known for the analysis and synthesis of synchronization dynamics of coupled and driven oscillators (Parekh *et al*., 1997; He and Vaidya, 1994). In addition to the familiar approach of analyzing and synthesizing the synchronization behavior of coupled systems; the present paper employed the Lyapunov direct method to obtain the threshold coupling at which the two systems become completely synchronized. To begin with, we have applied the following assumption to prove the main theorem of this paper.

Assumption. The chaotic trajectory of the master finance (1) is bounded i.e. for any bounded initial condition x(0) within the defining domain of the drive system, there exists a positive real constant, σ, such that |(x(t)| ≤ σ ∀t ≥ 0.

Remark 1. This assumption is reasonable and valid in the context of bounded feature of chaotic attractors (Curran and Chua, 1997).

Next, we proceed by utilizing the stability theory on time-varied systems (Liao and Wang, 2007) to derive sufficient criteria for global chaos synchronization in the sense of the error system (5). The following theorem is related to the general control matrix.

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \in \mathbb{R}^{3\times 3} \qquad (9)$$

Theorem 1. The master-slave system (2) achieves global chaos synchronization if a symmetric positive matrix.

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} \qquad (10)$$

And a coupling matrix K ∈ $\mathbb{R}^{3\times 3}$ defined in (8) are chosen such that for any $\mathrm{t} > 0$

$$\begin{aligned} &\Omega_1 < 0 \\ &\Omega_2 < 0 \\ &\Omega_3 < 0 \\ &4\Omega_1\Omega_2\Omega_2 > L^3 \end{aligned} \qquad (11)$$

Where $\Omega_1 = -k_{11}p_{11} - k_{21}p_{12} - ({x_1}^2 - {y_1}^2)p_{12} - p_{13} - k_{31}p_{13} + ({x_1}^2 - {y_1}^2 - a)p_{11}$, $\Omega_2 = -k_{22}p_{22} - k_{21}p_{12} - k_{32}p_{23} - bp_{22}$, $\Omega_3 = -k_{13}p_{13} - k_{23}p_{23} - k_{33}p_{33} - p_{13} - cp_{33}$ and $L^3 = \frac{1}{2}({\mu_{23}}^2\mu_{11} + {\mu_{12}}^2\mu_{33} + {\mu_{13}}^2\mu_{22}) - \mu_{12}\mu_{13}\mu_{23}$

Proof. Let us assume a quadratic Lyapunov function of the form:

$$V(e) = e^T P e \qquad (12)$$

Where P is a positive definite symmetric matrix defined in (10). The derivative of the Lyapunov function with respect to time, t, along the trajectory of the error system (5) is of the form.

$$\dot{V}(e) = \dot{e}^T P e + e^T P \dot{e} \qquad (13)$$

Substituting Eq. (5) into the system (12), we have

$$\dot{V}(e) = e^T[(A - K + M)^T P + P(A - K + M)]e \qquad (14)$$

$$\dot{V}(e) < 0, \text{ If } \lambda = (A - K + M)^T P + P(A - K + M) < 0 \qquad (15)$$

That is

$$\lambda = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_{22} & \mu_{23} \\ \mu_{13} & \mu_{23} & \mu_{33} \end{pmatrix} \qquad (16)$$

Where $\mu_{11} = -2(a + k_{11} - \alpha)p_{11} - 2(k_{21} + \beta)p_{12} - 2(1 + k_{31})p_{13}$; $\mu_{12} = -k_{12}p_{11} - (k_{11} + k_{22} + a + b - \alpha)p_{12} - k_{32}p_{13} - (k_{21} + \beta)p_{22} - (1 + k_{31})p_{23}$; $\mu_{13} = (1 - k_{13})p_{11} - k_{23}p_{12} - (k_{11} + k_{33} + a + c - \alpha)p_{13} - (k_{21} + \beta)p_{23} - (1 + k_{31})p_{33}$; $\mu_{22} = -2[k_{12}p_{12} + (b + k_{22})p_{22} + k_{32}p_{13}]$; $\mu_{23} = (1 - k_{13})p_{12} - k_{12}p_{13} - k_{23}p_{22} - (k_{22} + k_{33} + b + c)p_{23} - k_{32}p_{33}$; $\mu_{33} = 2[(1 - k_{13})p_{13} - k_{23}p_{23} - (k_{33} + c)p_{33}]$; $\alpha = x_1x_2 - y_1y_2$ and $\beta = {x_1}^2 - {y_1}^2$ respectively.

The symmetric matrix in (16) is negative definite if and only if

$$\begin{aligned}&-2(k_{11}+a-\alpha)p_{11}-2(k_{21}+\beta)p_{12}-2(1+k_{31})p_{13}<0\\&-2k_{12}p_{12}-2(k_{22}+b)p_{22}-2k_{32}p_{23}<0\\&2(1-k_{13})p_{13}-2k_{23}p_{23}-2(k_{33}+c)p_{33}<0\\&4L_1L_2L_3-L^3>0\end{aligned} \tag{17}$$

Where $L_2=-k_{22}p_{22}-k_{21}p_{12}-k_{32}p_{23}-bp_{22}$; $L_3=-k_{13}p_{13}-k_{23}p_{23}-k_{33}p_{33}+p_{13}-cp_{33}$; and $L^3=\frac{1}{2}({\mu_{23}}^2\mu_{11}+{\mu_{12}}^2\mu_{33}+{\mu_{13}}^2\mu_{22})-\mu_{12}\mu_{13}\mu_{23}$.

Since the matrix P is positive definite, we have $p_{11}(p_{22}p_{33}-{p_{23}}^2)-{p_{12}}^2p_{33}+2p_{12}p_{13}p_{23}-{p_{13}}^2p_{22}>0$, so that

$$-2p_{11}k_{11}-2(a-\alpha)p_{11}-2k_{21}p_{12}-2\beta p_{12}-2(1+k_{31})p_{13}\le -2p_{11}k_{11}-2ap_{11}-2p_{12}k_{21}-2(1+k_{31})p_{13}+2\alpha p_{11}+2\beta|p_{12}|\le 2\Omega_1 \tag{18}$$

$$|-p_{11}k_{12}-(k_{11}+k_{22})p_{12}-(a+b-\alpha)p_{12}-p_{13}k_{32}-p_{11}k_{11}-(1+k_{31})p_{23}-p_{22}k_{21}-\beta p_{22}|\le|-p_{11}k_{12}-(k_{11}+k_{22})p_{12}-(a+b)p_{12}-p_{13}k_{32}-p_{11}k_{11}-(1+k_{31})p_{23}-p_{22}k_{21}|+\alpha p_{12}-\beta p_{22} \tag{19}$$

$$|(1-k_{13})p_{11}-(k_{11}+k_{33}+a+c-\alpha)p_{13}-p_{12}k_{23}-p_{23}k_{21}-\beta p_{23}-(1+k_{31})p_{33}|\le \alpha p_{13}-\beta p_{23}+|(1-k_{13})p_{11}-p_{12}k_{23}-p_{23}k_{21}-k_{11}p_{13}-k_{33}p_{13}-(a+c)p_{13}-(1+k_{31})p_{33}| \tag{20}$$

The inequalities (11) hold if the inequalities (17) are satisfied. This completes the proof.

For the purpose of applications, it is necessary that the simplest possible synchronization controllers are employed. Hence, the following corollaries can be obtained from the main theorem of this paper.

Corollary 1: If the coupling matrix is defined by $K=diag\{k_1,k_2,k_3\}$ and the symmetric positive definite matrix P is as defined in (10) such that

$$\begin{aligned}&k_1>\frac{(\alpha-a)p_{11}-\beta p_{12}-p_{13}}{p_{11}}\\&k_2>-b\\&k_3>\frac{p_{13}-cp_{33}}{p_{33}}\end{aligned} \tag{21}$$

$$4[(\alpha-a-k_1)p_{11}-\beta p_{12}-p_{13}]\big(-p_{22}(b+k_2)\big)[p_{13}-(k_3+c)p_{33}]>[(p_{13}-(k_3+c)p_{33})((k_1+k_2+a+b-\alpha)p_{12}+\beta p_{22}+p_{23})^2]-[(k_1+a-\alpha)p_{11}+\beta p_{12}+p_{13}](p_{12}-p_{23}(k_2+k_3+b+c))^2+[(p_{12}-p_{23}(k_2+k_3+b+c))(p_{11}-(k_1+k_3+a+c-\alpha)p_{13}-\beta p_{23}-p_{33})((k_1+k_2+a+b-\alpha)p_{12}+\beta p_{22}+p_{23})]+(p_{11}-(k_1+k_3+a+c-\alpha)p_{13}-\beta p_{23}-p_{33})^2(p_{22}(b+k_2)) \tag{22}$$

Then, the master-slave system (2) achieves global chaos synchronization.

Proof. The inequalities (21) can be obtained according to the inequalities (11) with $k_{11}=k_1, k_{22}=k_2, k_{33}=k_3$ and $k_{12}=k_{13}=k_{21}=k_{23}=k_{31}=k_{32}=0$.

Corollary 2: The master-slave system (2) achieves global chaos synchronization if the coupling matrix **K** = diag (k,k,k) and the positive symmetric matrix **P** defined in (10) are chosen such that

$$k=max\left(\frac{(\alpha-a)p_{11}-\beta p_{12}-p_{13}}{p_{11}},-b,\frac{p_{13}-cp_{33}}{p_{33}}\right)\ge 0 \tag{23}$$

$$4(p_{11}p_{22}p_{33}-p_{11}{p_{23}}^2-p_{22}{p_{13}}^2-p_{33}{p_{12}}^2+2p_{12}p_{13}p_{23})k^3+4(bp_{11}p_{22}p_{33}-bp_{22}{p_{13}}^2+\omega_1{p_{12}}^2-\omega_5p_{12}p_{13}+\omega_2p_{22}p_{33}-\omega_3p_{12}p_{33}-\omega_2{p_{23}}^2-\omega_4p_{12}p_{23}-\omega_1p_{11}p_{22}+\omega_4p_{22}p_{13}+\omega_5p_{23}p_{11}+\omega_3p_{13}p_{23})k^2+4\left(\omega_2\omega_5p_{23}-\omega_1\omega_2p_{22}+b\omega_2p_{22}p_{33}-b\omega_1p_{11}p_{22}+b\omega_4p_{22}p_{13}-\frac{\omega_3\omega_5}{2}p_{13}-\frac{{\omega_4}^2}{4}p_{22}+\omega_1\omega_3p_{12}-\frac{{\omega_5}^2}{4}p_{11}+\frac{\omega_4\omega_5}{2}p_{12}-\frac{\omega_3\omega_4}{2}p_{23}-\frac{{\omega_3}^2}{4}p_{33}\right)k+4\left(\frac{\omega_3\omega_4\omega_5}{4}-b\omega_1\omega_2p_{22}+\frac{\omega_1{\omega_3}^2}{4}-b\frac{{\omega_4}^2}{4}p_{22}-\frac{\omega_2{\omega_5}^2}{4}\right)<0 \tag{24}$$

Where $\omega_1 = p_{13} - cp_{33}$, $\omega_2 = (a-\alpha)p_{11} + \beta p_{12} + p_{13}$; $\omega_3 = (a+b-\alpha)p_{12} + \beta p_{22} + p_{23}$; $\omega_4 = p_{11} - (a+c-\alpha)p_{13} - \beta p_{23} - p_{33}$; $\omega_5 = p_{12} - p_{23}(b+c)$.

Proof. Letting $k_1 = k_2 = k_3 = k$ in the partial synchronization conditions ((21), the inequality (23) can be obtained. For $k > 0$ given by (23) we have

$$\begin{aligned} &[\left|-p_{23} - \left(2k - (a+b-\alpha)\right)p_{12}\right| - \beta p_{22}]^2 \le [\left|-p_{23} - \left((a+b-\alpha)\right)p_{12}\right| + 2k|p_{12}| - \beta p_{22}]^2; \\ &[|p_{11} - (2k+a+c-\alpha)p_{13}| - \beta p_{23} - p_{33}]^2 \le [|p_{11} - (a+c-\alpha)p_{13}| + 2k|p_{13}| - \beta p_{23} - p_{33}]^2; \\ &|p_{12} - (2k+b+c)p_{23}|]^2 \le [|p_{12} - (b+c)p_{23}| + 2k|p_{23}|]^2 \end{aligned} \quad (25)$$

Hence, the inequality (24) can be realized by partial synchronization criterion ((22)) with $k_1 = k_2 = k_3 = k$. Since $p_{11}(p_{22}p_{33} - {p_{23}}^2) - {p_{12}}^2 p_{33} + 2p_{12}p_{13}p_{23} - {p_{13}}^2 p_{22} > 0$, the solution k to the inequality (24) exists.

Remark 2. We select the elements of the positive symmetric matrix **P** as $p_{12} = p_{13} = p_{23} = 0$, $p_{11} = \beta p_{22} = p_{33}$, and obtain the following algebraic synchronization criterion via the inequalities (23) and (24).

$$\begin{aligned} &k = diag\{k,k,k\} \\ &k > \frac{(a+b-\alpha) + \sqrt{(\alpha-a+b)^2 + \beta}}{2} = k_{th} \end{aligned} \quad (26)$$

Corollary 3: The synchronization scheme (3) achieves global chaos synchronization, if the control matrix $K = diag\{k,0,0\}$ and a symmetric positive definite matrix **P** given in (10) are selected such that

$$\begin{aligned} &k > \frac{(\alpha-a)p_{11} - \beta p_{12} - p_{13}}{p_{11}} \\ &-bp_{22} < 0 \\ &p_{13} - cp_{33} < 0 \end{aligned} \quad (27)$$

$$k^2({p_{12}}^2\omega_1 + bp_{22}{p_{13}}^2) + k(2\omega_1\omega_3 p_{12} - {\omega_5}^2 p_{11} - \omega_5 p_{13} - 2b\omega_4 p_{13}p_{22} - 4b\omega_1 p_{11}p_{22}) + (\omega_1{\omega_3}^2 - \omega_2{\omega_5}^2 + b{\omega_4}^2 p_{22} - 4b\omega_1\omega_2 p_{22} + \omega_4\omega_5) < 0 \quad (28)$$

Where $\omega_1 = p_{13} - cp_{33}$, $\omega_2 = (a-\alpha)p_{11} + \beta p_{12} + p_{13}$; $\omega_3 = (a+b-\alpha)p_{12} + \beta p_{22} + p_{23}$; $\omega_4 = p_{11} - (a+c-\alpha)p_{13} - \beta p_{23} - p_{33}$; $\omega_5 = p_{12} - p_{23}(b+c)$

Remark 3. We select the symmetric positive definite matrix

$\boldsymbol{P} = p_{22}\begin{pmatrix} \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}$**;** with $p_{22} > 0$.

The following synchronization criterion is gained based on the criteria (26) through (28).

$$\begin{aligned} &k = diag\{k,0,0\} \\ &k > \ k > \frac{4b(a-\alpha) - \beta}{4b} \end{aligned} \quad (29)$$

Hyperchaotic finance system

Considering the new hyperchaotic finance system as defined by a set of four-order differential equations as follows:

$$\begin{aligned} &\dot{x}_1 = x_3 + (x_2 - a)x_1 + x_4 \\ &\dot{x}_2 = 1 - bx_2 - {x_1}^2 \\ &\dot{x}_3 = -x_1 - cx_3 \\ &\dot{x}_4 = -dx_1x_2 - ex_4 \end{aligned} \quad (30)$$

Where $a = 0.9, b = 0.2, c = 1.2, d = 0.2, e = 0.17$ where $x = [x_1, x_2, x_3, x_4]^T \epsilon \mathbb{R}^4$ are state space variables and a, b, c, d and e are positive real constants, they represent the interest rate, investment demand, price exponent, per investment cost and elasticity of demands of commercials respectively. The nonlinear hyper-chaotic finance given by Eq. (30) exhibits varieties of dynamical behavior including chaotic motion for the following parameter values where $a = 0.9, b = 0.2, c = 1.5,\ d = 0.2,\ e = 0.17$ as given in (Chen, 2002).

To facilitate the present analysis, we express system (30) in the following vector form:

$$\dot{x} = Ax - f(x) + G(x) \tag{31}$$

In order to examine the synchronization between two unidirectional coupled hyper-chaotic finances, we construct a master-slave synchronization scheme for two identical hyper-chaotic finance by linear state error feedback controller in the following form:

$$\begin{aligned} M &: \dot{x} = Ax - f(x) + G(x) \\ S &: \dot{y} = Ay - f(y) + G(y) + u(t) \\ C &: u(t) = K(x - y), \end{aligned} \tag{32}$$

Where u = K(x−y) is the linear state feedback control input and $K \in \mathbb{R}^{4\times4}$ is a constant control matrix that determines the strength of the feedback into the response system. By defining the synchronization error variable as the difference between the relevant dynamical variables given by:

$$e = x - y \tag{33}$$

We obtain the error dynamics for the master-slave system ((32) as:

$$\dot{e} = \big(A - K + M(x,y)\big)e \tag{34}$$

Where $M(x,y) = \begin{pmatrix} x_1x_2 - y_1y_2 \\ -({x_1}^2 - {y_1}^2) \\ 0 \\ -d(x_1x_2 - y_1y_2) \end{pmatrix}$ $and\ G(x,y) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

The hyper-chaotic finance system (30) has three equilibrium points:

$\rho_1 = \left(0, \frac{1}{b}, 0,0\right)$; $\rho_{2,3} = \left(\pm\varepsilon, \frac{e+ace}{c(e-d)}, \mp\frac{\varepsilon}{c}, \frac{d\varepsilon(1+ac)}{c(d-e)}\right)$; Where $\varepsilon = \sqrt{1 + \frac{eb+abce}{c(d-e)}}$

Thus, the Jaccobian matrix is given by

$$J = \begin{vmatrix} (x_2 - a) & x_1 & 1 & 1 \\ -2x_1 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ -dx_2 & -dx_1 & 0 & -e \end{vmatrix} \tag{35}$$

Then, the eigenvalues for the respective equilibrium points at $a = 0.9, b = 0.2, c = 1.2, d = 0.2, e = 0.17$, are given by: ($\lambda_{11} = 3.6298$, $\lambda_{21} = 0.1412, \lambda_{31} = -1.0410$ $\lambda_{41} = -0.2000$); ($\lambda_{12} = -10.2024$, $\lambda_{22} = 0.0209$, $\lambda_{32} = -0.7168$ $\lambda_{42} = -1.393$) and ($\lambda_{13} = -10.2024$, $\lambda_{23} = 0.0209$, $\lambda_{33} = -0.7168$, $\lambda_{43} = -1.3939$) respectively.

In the absence of the control matrix, K, Eq. (34), would have three equilibrium points at $(0,5,0,0)$, $(\pm1.7218, -9.8222, \mp1.4348,\ 19.8958)$. Our aim is to choose the appropriate coupling matrix K, such that the trajectories of the master system, $x(t)$, and slave one $y(t)$ satisfy

$$\lim_{t\to\infty} \parallel e \parallel = \lim_{t\to\infty} \parallel x(t) - y(t) \parallel = 0 \tag{36}$$

Where || $*$ || represents Euclidean norm of a vector.

Assumption. The chaotic trajectory of the master hyper-chaotic finance (30) is bounded i.e. for any bounded initial condition x(0) within the defining domain of the drive system, there exists a positive real constant, σ, such that $|(x(t)| \leq \sigma\ \forall t \geq 0$.

Remark 1 This assumption is reasonable and valid in the context of bounded feature of chaotic attractors (Curran and Chua, 1997).

Next, we proceed by utilizing the stability theory on time-varied systems (Liao and Wang, 2007) to derive sufficient criteria for global chaos synchronization in the sense of the error system (33). The following theorem is related to the general control matrix.

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix} \in \mathbb{R}^{4\times4} \quad (37)$$

Theorem 1. The master-slave system (30) achieves global chaos synchronization if a symmetric positive matrix.

$$\boldsymbol{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & p_{33} & p_{34} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{pmatrix} \quad (38)$$

And a coupling matrix **K** $\in \mathbb{R}^{4\times4}$ defined in (37) are chosen such that for any t > 0

$$\begin{aligned} &\Phi_1 = -p_{11}(a-\alpha+k_{11}) - p_{12}k_{21} - p_{13}(1+k_{31}) - p_{14}(d\alpha+k_{41}) - |p_{12}|\beta < 0 \\ &\Phi_2 = -p_{12}k_{12} - p_{22}(b+k_{22}) - p_{23}k_{32} - p_{24}k_{42} < 0 \\ &\Phi_3 = -p_{13}k_{13} + p_{13} - p_{23}k_{23} - p_{33}(c+k_{33}) - p_{34}k_{34} < 0 \\ &\Phi_4 = p_{14}(1-k_{14}) - p_{24}k_{24} - p_{34}k_{34} - p_{44}(e+k_{44}) \\ &\text{such that } 4\Phi_1\Phi_2\Phi_3\Phi_4 = L^4 \end{aligned} \quad (39)$$

Where $L^4 = \frac{1}{4}[(\mu_{12}{}^2\mu_{34}{}^2 - \mu_{11}\mu_{23}{}^2\mu_{44} + \mu_{13}{}^2\mu_{24}{}^2 - \mu_{11}\mu_{24}{}^2\mu_{33} + 2\mu_{11}\mu_{23}\mu_{24}\mu_{34} - \mu_{13}{}^2\mu_{22}\mu_{44} + \mu_{12}\mu_{13}\mu_{23}\mu_{44} - \mu_{12}\mu_{14}\mu_{23}\mu_{34} + \mu_{14}{}^2\mu_{23}{}^2 - \mu_{11}\mu_{22}\mu_{34}{}^2 + 2\mu_{12}\mu_{14}\mu_{24}\mu_{33} - \mu_{12}{}^2\mu_{33}\mu_{44} + \mu_{12}\mu_{13}\mu_{23}\mu_{24} - 2\mu_{12}\mu_{13}\mu_{24}\mu_{34} + 2\mu_{13}\mu_{14}\mu_{22}\mu_{34} - 2\mu_{13}\mu_{14}\mu_{23}\mu_{24} - \mu_{14}{}^2\mu_{22}\mu_{33} - \mu_{12}\mu_{14}\mu_{23}\mu_{24})]$

Proof. Let us assume a quadratic Lyapunov function of the form:

$$V(e) = e^T P e \quad (40)$$

Where $\boldsymbol{P}$ is a positive definite symmetric matrix defined in (38). The derivative of the Lyapunov function with respect to time, t, along the trajectory of the error system (33) is of the form:

$$\dot{V}(e) = \dot{e}^T P e + e^T P \dot{e} \quad (41)$$

Substituting Eq. (34) into the system (41), we have

$$\dot{V}(e) = e^T[(A-K+M)^T P + P(A-K+M)]e \quad (42)$$

$\dot{V}(\boldsymbol{e}) < 0$, if

$$\lambda = (A-K+M)^T P + P(A-K+M) < 0 \quad (43)$$

That is, $$\lambda = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{12} & \mu_{22} & \mu_{23} & \mu_{24} \\ \mu_{13} & \mu_{23} & \mu_{33} & \mu_{34} \\ \mu_{14} & \mu_{24} & \mu_{34} & \mu_{44} \end{pmatrix} \quad (44)$$

Where $\mu_{11} = -2p_{11}(a+k_{11}-\alpha) - 2p_{12}(k_{21}+\beta) - 2p_{13}(1+k_{31}) - 2p_{14}(k_{41}+d\alpha)$; $\mu_{12} = -p_{11}k_{12} - p_{12}(k_{11}+k_{22}+a+b-\alpha) - p_{13}k_{32} - p_{14}k_{42} - p_{22}(k_{21}+\beta) - p_{23}(k_{31}+1) - p_{24}(d\alpha+k_{41})$;
$\mu_{13} = p_{11}(1-k_{13}) - p_{12}k_{23} - p_{13}(k_{11}+k_{33}+a+c-\alpha) - p_{14}k_{43} - p_{23}(\beta+k_{21}) - p_{33}(1+k_{31}) - p_{34}(d\alpha+k_{41})$; $\mu_{14} = p_{11}(1-k_{14}) - p_{12}k_{24} - p_{13}k_{34} - p_{14}(k_{11}+k_{44}+a+e-\alpha) - p_{24}(\beta+k_{21}) - p_{34}(1+k_{31}) - p_{44}(d\alpha+k_{41})$; $\mu_{22} =$

$-2[p_{12}k_{12}+p_{22}(b+k_{22})+p_{23}k_{32}+p_{24}k_{42}$; $\mu_{23}=p_{12}(1-k_{13})-p_{13}k_{12}-p_{22}k_{23}-p_{23}(k_{22}+k_{33}+b+c)-p_{24}k_{43}-p_{33}k_{32}-p_{34}k_{42}$; $\mu_{24}=p_{12}(1-k_{14})-p_{14}k_{12}-p_{23}k_{34}-p_{24}(k_{22}+k_{44}+b+e)-p_{44}k_{42}$; $\mu_{33}=2[p_{13}(1-k_{13})-p_{23}k_{23}-p_{33}(c+k_{33})-p_{34}k_{43}$; $\mu_{34}=p_{13}(1-k_{14})+p_{14}(1-k_{13})-p_{23}k_{24}-p_{24}k_{23}-p_{33}k_{34}-p_{34}(k_{33}+k_{44}+c+e)-p_{44}k_{43}$ and $\mu_{44}=2[p_{14}(1-k_{14})-p_{24}k_{24}--p_{34}k_{34}-p_{44}(e+k_{44})]$.

The symmetric matrix in (44) is negative definite if and only if

$$\begin{aligned}&-2p_{11}k_{11}-2p_{11}(a-\alpha)-2p_{12}(k_{21}+\beta)-2p_{13}(1+k_{31})-2p_{14}(k_{41}+d\alpha)<0\\&-2[p_{12}k_{12}+p_{22}(b+k_{22})+p_{23}k_{32}+p_{24}k_{42}]<0\\&2[p_{13}(1-k_{13})-p_{23}k_{23}-p_{33}(c+k_{33})-p_{34}k_{43}]<0\\&2p_{14}(1-k_{14})-2p_{24}k_{24}-2p_{34}k_{34}-2p_{44}(e+k_{44})<0\\&4\mathrm{L}_1\mathrm{L}_2\mathrm{L}_3\mathrm{L}_4-L_5>0\end{aligned} \tag{45}$$

Where $L_1=|-p_{12}(k_{21}+\beta)-p_{13}(1+k_{31})-2p_{14}(k_{41}+d\alpha)-p_{11}k_{11}|$; $L_2=|p_{12}k_{12}+p_{22}(b+k_{22})+p_{23}k_{32}+p_{24}k_{42}|$; $L_3=|p_{13}(1-k_{13})-p_{23}k_{23}-p_{33}(c+k_{33})-p_{34}k_{43}|$; $L_4=|p_{14}(1-k_{14})-p_{24}k_{24}-p_{34}k_{34}-p_{44}(e+k_{44})|$ and $L_5=L^4$.

Since the matrix **P** is positive definite, we have

$$p_{11}p_{22}p_{33}p_{44}-\mathcal{P}^4>0 \tag{46}$$

Where $\mathcal{P}^4=(p_{12}{}^2p_{34}{}^2-p_{11}p_{23}{}^2p_{44}+p_{13}{}^2p_{24}{}^2-p_{11}p_{24}{}^2p_{33}+2p_{11}p_{23}p_{24}p_{34}-p_{13}{}^2\mu_{22}\mu_{44}+p_{12}p_{13}p_{23}p_{44}-p_{12}p_{14}p_{23}p_{34}+p_{14}{}^2p_{23}{}^2-p_{11}p_{22}p_{34}{}^2+2p_{12}p_{14}p_{24}p_{33}-p_{12}{}^2p_{33}p_{44}+p_{12}p_{13}p_{23}p_{24}-2p_{12}p_{13}p_{24}p_{34}+2p_{13}p_{14}p_{22}p_{34}-2p_{13}p_{14}p_{23}p_{24}-p_{14}{}^2p_{22}p_{33}-p_{12}p_{14}p_{23}p_{24})$; so that $-2p_{11}(a-\alpha+k_{11})-2p_{12}k_{21}-2p_{13}(1+k_{31})-2p_{14}(d\alpha+k_{41})-2p_{12}\beta\le-2p_{11}k_{11}-2p_{11}(a-\alpha)-2p_{12}k_{21}-2p_{13}(1+k_{31})-2p_{14}(d\alpha+k_{41})+2|p_{12}|\beta\le2\Phi_1$, $|-p_{11}k_{12}-p_{12}(k_{11}+k_{22}+a+b-\alpha)-p_{13}k_{32}-p_{14}k_{42}-p_{22}(k_{21}+\beta)-p_{23}(k_{31}+1)-p_{24}(d\alpha+k_{41}|\le|-p_{11}k_{12}-p_{12}(k_{11}+k_{22}+a+b-\alpha)-p_{13}k_{32}-p_{14}k_{42}-p_{22}k_{21}-p_{23}(k_{31}+1)-p_{24}(d\alpha+k_{41}|-p_{22}\beta$, $|p_{11}(1-k_{13})-p_{12}k_{23}-p_{13}(k_{11}+k_{33}+a+c-\alpha)-p_{14}k_{43}-p_{23}(\beta+k_{21})-p_{33}(1+k_{31})-p_{34}(d\alpha+k_{41})|\le|p_{11}(1-k_{13})-p_{12}k_{23}-p_{13}(k_{11}+k_{33}+a+c-\alpha)-p_{14}k_{43}-p_{23}k_{21}-p_{33}(1+k_{31})-p_{34}(d\alpha+k_{41})|-p_{23}\beta$, $|p_{11}(1-k_{14})-p_{12}k_{24}-p_{13}k_{34}-p_{14}(k_{11}+k_{44}+a+e-\alpha)-p_{24}(\beta+k_{21})-p_{34}(1+k_{31})-p_{44}(d\alpha+k_{41})|\le|p_{11}(1-k_{14})-p_{12}k_{24}-p_{13}k_{34}-p_{14}(k_{11}+k_{44}+a+e-\alpha)-p_{24}k_{21}-p_{34}(1+k_{31})-p_{44}(d\alpha+k_{41})|-\beta p_{24}$,

The inequalities (45)) hold if the inequalities (39) are satisfied. This completes the proof.

For the purpose of applications, it is necessary that the simplest possible synchronization controllers are employed. Hence, the following corollaries can be obtained from the main theorem of this paper.

Corollary 1: If the coupling matrix is defined by $K=diag\{k_1,k_2,k_3,k_4\}$ and the symmetric positive definite matrix **P** is as defined in (45) such that

$$\begin{aligned}&k_1>\frac{(\alpha-a)p_{11}+\beta|p_{12}|-p_{13}}{p_{11}}\\&k_2>-b\\&k_3>\frac{p_{13}-cp_{33}}{p_{33}}\\&k_4>\frac{p_{14}-ep_{44}}{p_{44}}\end{aligned} \tag{47}$$

$$4[|p_{22}|(b+k_2)(k_1p_{11}+\varphi_2)(k_3p_{33}-\varphi_1)(k_4p_{44}-\varphi_3)]>[p_{12}(k_1+k_2)+\varphi_4]^2(k_3p_{33}-\varphi_1)(k_4p_{44}-\varphi_3)+[p_{13}(k_1+k_3)-\varphi_5]^2\big(p_{22}(k_2+b)\big)(k_4p_{44}-\varphi_3)+[p_{14}(k_1+k_4)-\varphi_6]^2\big(p_{22}(k_2+b)\big)(k_3p_{33}-\varphi_1) \tag{48}$$

Where $\varphi_1=p_{13}-cp_{33}$; $\varphi_2=p_{11}(a-\alpha)+p_{12}\beta+p_{13}+p_{14}d\alpha$; $\varphi_3=p_{14}-p_{44}e$; $\varphi_4=p_{12}(a+b-\alpha)+p_{22}\beta+p_{23}+p_{24}d\alpha$; $\varphi_5=p_{11}-p_{13}(a+c-\alpha)-p_{23}\beta-p_{33}-p_{34}d\alpha$.

By ignoring some terms in order to reduce stress Eq. (48) is obtained. Thus, the master-slave system (32) achieves global chaos synchronization.

Proof: The inequalities (47) can be obtained according to the inequalities (39) with $k_{11}=k_1,k_{22}=k_2,k_{33}=k_3,k_{44}=k_4$ and $k_{12}=k_{13}=k_{14}=k_{21}=k_{23}=k_{24}=k_{31}=k_{32}=k_{34}=0$.

Corollary 2: The master-slave system (32) achieves global chaos synchronization if the coupling matrix $\boldsymbol{K} = diag\{k,k,k,k\}$ and the positive symmetric matrix **P** defined in (38) are chosen such that

$$k = max\left(\frac{(\alpha-a)p_{11}-\beta p_{12}-p_{13}}{p_{11}}, -b, \frac{p_{13}-cp_{33}}{p_{33}}, \frac{p_{14}-ep_{44}}{p_{44}}\right) \geq 0 \tag{49}$$

$$\boldsymbol{4k^4} + 4k^3(a+b+c+e-\alpha) + 4k^2\Big[b(a-\alpha)+c(a-\alpha)+e(a-\alpha)+bc+be+ce-\frac{p_{11}}{4p_{44}}-\frac{p_{44}}{p_{11}}(\frac{d\alpha}{2})^2+\left(\frac{d\alpha}{2}\right)-\frac{p_{11}}{4p_{33}}-\frac{p_{33}}{4p_{11}}+\frac{1}{2}-\frac{p_{22}}{p_{11}}\left(\frac{\beta}{2}\right)^2\Big] + 4k\Big[bce+(a-\alpha)(bc+be+ce)-\frac{p_{11}}{p_{44}}\left(\frac{b+c}{4}\right)-\frac{p_{44}}{p_{11}}(b+c)(\frac{d\alpha}{2})^2+(b+c)\left(\frac{d\alpha}{2}\right)-\frac{(b+e)p_{11}}{4p_{33}}-\frac{(b+e)p_{33}}{4p_{11}}+\left(\frac{b+e}{2}\right)-\frac{p_{22}}{p_{11}}(c+e)\left(\frac{\beta}{2}\right)^2\Big] + 4\Big[bce(a-\alpha)-\frac{bcp_{11}}{4p_{44}}-bc\frac{p_{44}}{p_{11}}(\frac{d\alpha}{2})^2+\frac{bcd\alpha}{2}-be\frac{p_{11}}{4p_{33}}-\frac{p_{33}}{4p_{11}}be+\frac{be}{2}-ce\frac{p_{22}}{p_{11}}(\frac{\beta}{2})^2\Big] > 0 \tag{50}$$

Proof: Letting $k_1 = k_2 = k_3 = k_4 = k$ in the partial synchronization conditions (47), the inequality (48) can be obtained.

For $k > 0$ given by (48), we have

$$[|p_{12}(2k+a+b-\alpha)+p_{23}+p_{24}d\alpha|+p_{22}\beta]^2 \leq [|p_{12}(a+b-\alpha)+p_{23}+p_{24}d\alpha|+2k|p_{12}|+p_{22}\beta\,]^2$$
$$[|p_{11}-p_{13}(2k+a+c-\alpha)-p_{33}|-p_{23}\beta]^2 \leq [|p_{11}-p_{13}(a+c-\alpha)-p_{33}|+2k|p_{13}|-p_{23}\beta]^2$$
$$(|p_{11}-p_{14}(k_{11}+k_{44}+a+e-\alpha)-p_{34}-p_{44}d\alpha|-p_{24}\beta)^2 \leq (|p_{11}-p_{14}(a+e-\alpha)-p_{34}-p_{44}d\alpha|+2k|p_{14}|-p_{24}\beta)^2$$

Thus, the inequality (49) can be realized by partial synchronization criterion (48) with $k_1 = k_2 = k_3 = k_4 = k$.

Since $p_{11}p_{22}p_{33}p_{44} - [p_{12}{}^2p_{33}p_{44} + p_{13}{}^2p_{22}p_{44} + p_{14}{}^2p_{22}p_{33} + p_{23}{}^2p_{11}p_{44} + p_{24}{}^2p_{11}p_{33} + p_{34}{}^2p_{11}p_{22} - p_{12}{}^2p_{34}{}^2 - p_{13}{}^2p_{24}{}^2 - p_{14}{}^2p_{23}{}^2 - 2p_{11}p_{23}p_{24}p_{34} - 2p_{12}p_{14}p_{24}p_{33} - 2p_{13}p_{14}p_{22}p_{34} + 2p_{12}p_{13}p_{24}p_{34} + 2p_{13}p_{14}p_{23}p_{24} - p_{12}p_{13}p_{23}p_{24} + p_{12}p_{14}p_{23}p_{24} - p_{12}p_{13}p_{23}p_{24} + p_{12}p_{14}p_{23}p_{34}] > 0$,

Hence, the solution k to the inequality (50) exists.

Remark 2. We select the elements of the positive symmetric matrix **P** as $p_{12} = p_{13} = p_{14} = p_{23} = p_{24} = p_{34} = 0$, $p_{11} = \beta p_{22} = p_{33} = d\alpha p_{44}$, and obtain the following algebraic synchronization criterion via the inequalities (49) and (50).

$$k = diag\{k,k,k,k\}$$
$$k > \frac{(a+b-\alpha)+\sqrt{(\alpha-a+b)^2+\beta}}{2} = k_{th} \tag{51}$$

Corollary 3: The synchronization scheme (32) achieves global chaos synchronization, if the control matrix $K = diag\{k,0,0,0\}$ and a symmetric positive definite matrix **P** given in (38) are selected such that

$$\begin{aligned} k_1 &> \frac{(\alpha-a)p_{11}+\beta|p_{12}|-p_{13}}{p_{11}} \\ k_2 &> -b \\ k_3 &> \frac{p_{13}-cp_{33}}{p_{33}} \\ k_4 &> \frac{p_{14}-ep_{44}}{p_{44}} \end{aligned} \tag{52}$$

$$k^2(p_{14}{}^2b\varphi_1p_{22}+p_{13}{}^2b\varphi_3p_{22}-p_{12}{}^2\varphi_1\varphi_3)+k(b\varphi_1\varphi_3p_{11}p_{22}-2\varphi_1\varphi_3\varphi_4p_{12}-2b\varphi_3\varphi_5p_{13}p_{22}-2b\varphi_1\varphi_6p_{14}p_{22})+(b\varphi_1\varphi_2\varphi_3p_{22}-b\varphi_1\varphi_6{}^2p_{22}) < 0 \tag{53}$$

Where $\varphi_1 = p_{13} - cp_{33}$; $\varphi_2 = p_{11}(a-\alpha) + p_{12}\beta + p_{13} + p_{14}d\alpha$; $\varphi_3 = p_{14} - p_{44}e$, $\varphi_4 = p_{12}(a+b-\alpha) + p_{22}\beta + p_{23} + p_{24}d\alpha$; $\varphi_5 = p_{11} - p_{13}(a+c-\alpha) - p_{23}\beta - p_{33} - p_{34}d\alpha$; and $\varphi_6 = p_{11} - p_{14}(a+e-\alpha) - p_{24}\beta - p_{34} - p_{44}d\alpha$ respectively.

Remark 3. We select the symmetric positive definite matrix

$$\boldsymbol{P} = p_{22}\begin{pmatrix}\beta & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \beta & 0\\ 0 & 0 & 0 & \gamma\end{pmatrix},$$ where $\gamma = \frac{\beta}{d\alpha}$ with $p_{22} > 0$.

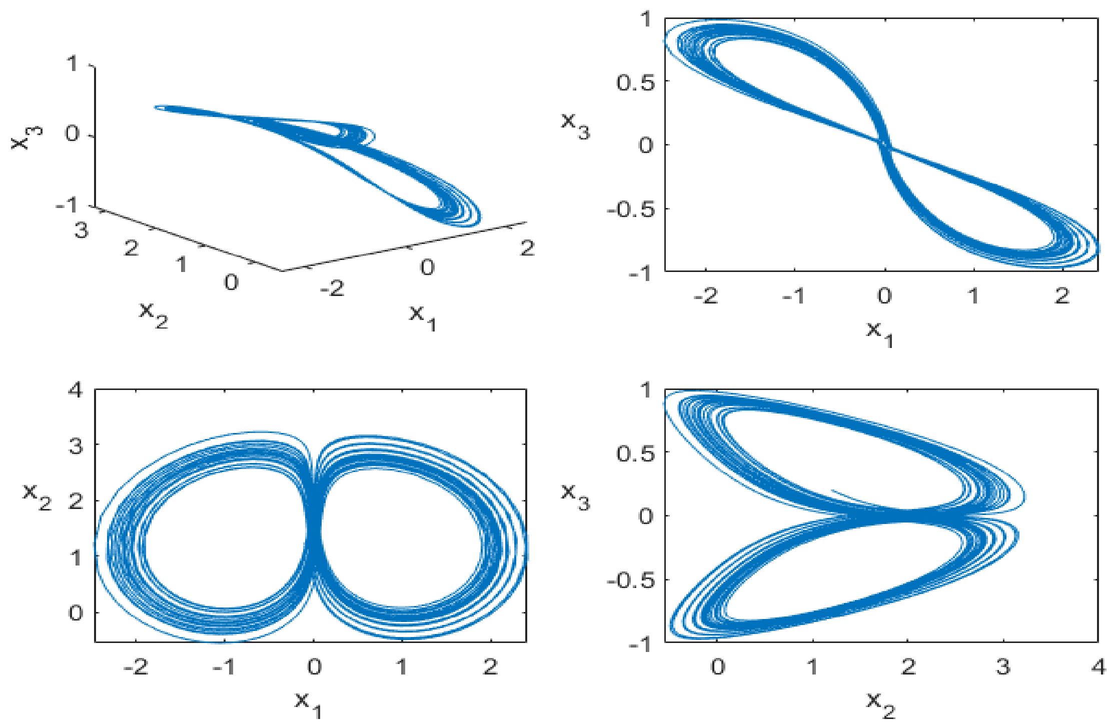

Figure 1. The phase portrait showing a chaotic attractor of nonlinear finance system with the following parameters $a = 0.8, b = 0.2, and\ \ c = 1.9$

The following synchronization criterion is gained based on the criteria (51) and (52).

$$k = diag\{k, 0,0,0\}$$
$$k > k > a - \alpha - \frac{\beta}{4b} \quad (54)$$

Where $\alpha = x_1x_2 - y_1y_2$ and $\beta = {x_1}^2 - {y_1}^2$

RESULTS AND DISCUSSION

In this section, we present numerical simulation results to confirm the obtained criteria for 3D and 4D finance systems.

Deduction from finance 3D system

Here, we utilized the fourth order Runge-Kutta routine with the following initial conditions: $(x_1(0), y_1(0)) = (-0.2, -0.1)$, $(x_2(0), y_2(0)) = (0.1, -0.1)$, $(x_3(0), y_3(0)) = (0.2, 0.1)$, a time-step of 0.001 and fixing the parameter values of $a = 0.9$, $b = 0.2$, $c = 1.7$ as in Figure 1, to ensure chaotic motion, we solved the master-slave system (2) with the control matrices as defined in Eqs. (26) and (29). The simulation results obtained reveal that the trajectory of the master finance system depicted in Figure 1, is bounded, the time series of finance system is shown in Figure 4 and the error dynamics shown in Figure 5 oscillate chaotically with time when the two systems are decoupled. The partial variables x_1, x_2 and x_3 of the chaotic attractor satisfy $x_1(\mathrm{t}) = -cx_3(\mathrm{t}) < 1$ for any t ≥ 0.

The critical coupling at which complete synchronization could be observed is vital for many scientific and technological applications because it provides useful information regarding the operational regime for optimal performance in coupled systems. In Figure 6, we displayed a simulation result of average error, E_{ave}, against coupling, k, and noticed that as k, increases and as full synchronization is approached, $E_{ave} \rightarrow 0$ asymptotically at the threshold coupling, $k_{th} \approx 0.9$. Then for all $k > k_{th}$, E_{ave} = 0 and remains stable as t → ∞ implying that the oscillators are completely synchronized. Interestingly, it was noticed that by direct calculations of Eq. (26) for the control matrix, $K = diag\{k, k, k\}$, $k > k_{th} = 1.0$. Thus, the obtained criterion is in good agreement with numerical simulation result. Using the criterion defined by Eq. (26), one readily obtains a coupling matrix $K = diag\{1.0,1.0,1.0\}$ by which the master-slave system (2) achieves chaos synchronization. Figure 7 shows the synchronization for $k = 1.2$. Finally, we depict the simulation results for the second case in which we choose constant control matrix $K = diag\{k, 0,0\}$, such that $k > 1.0$ which satisfies the condition in Eq. (29). The simulation results displayed in Figure 7 confirmed that complete synchronization is achieved for $k = 5.0 > k_{th}$. Notice that in both cases, the synchronization is already reached at t = 25, showing an excellent transient performance.

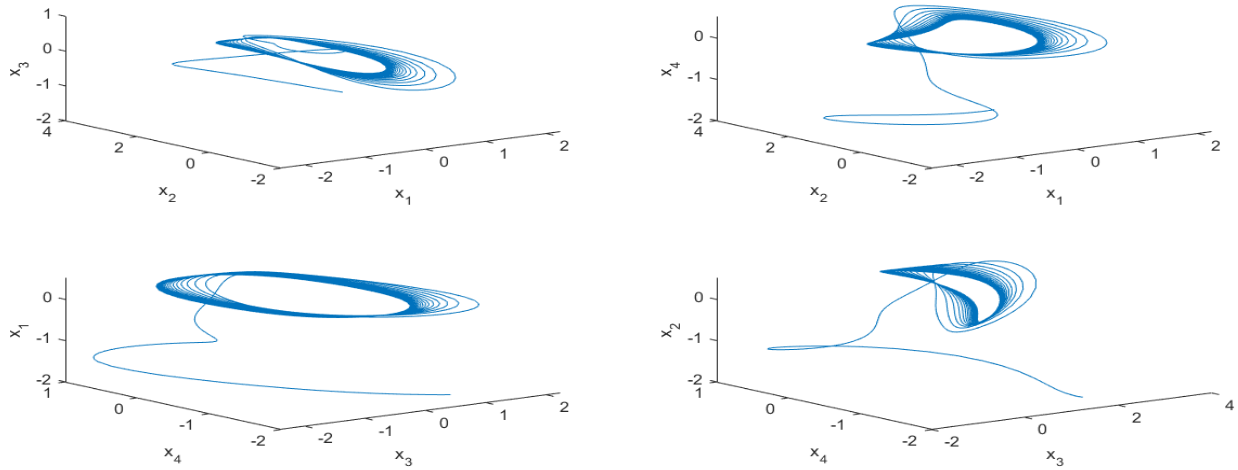

Figure 2. The phase portrait showing a chaotic attractor of nonlinear hyperchaotic finance system with the following parameters: $a = 0.9, b = 0.2, c = 1.5, d = 0.2$ *and* $e = 0.17$.

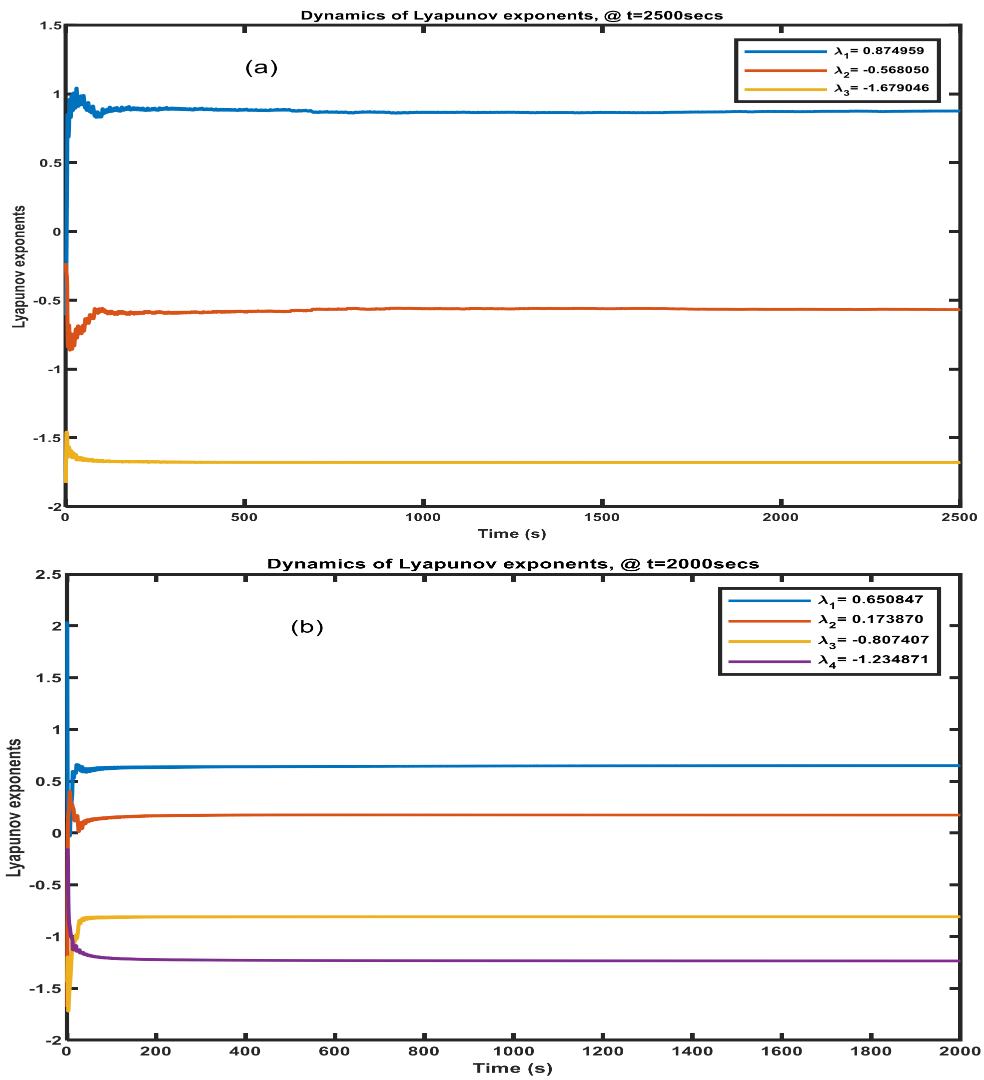

Figure 3. Showing the dynamics of Lyapunov exponents for (a) finance system (b) hyperchaotic finance system.

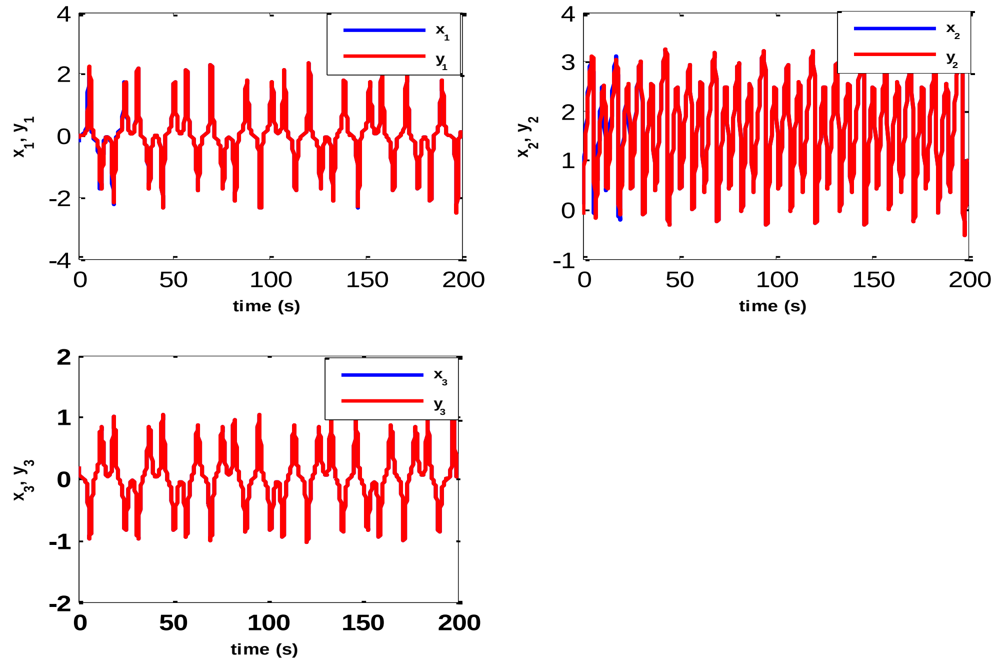

Figure 4. Time series for the synchronized master-slave systems $x = (-0.2, 0.1, 0.2)$ and $y = (-0.1, -0.1, 0.1)$ respectively with the designed controllers.

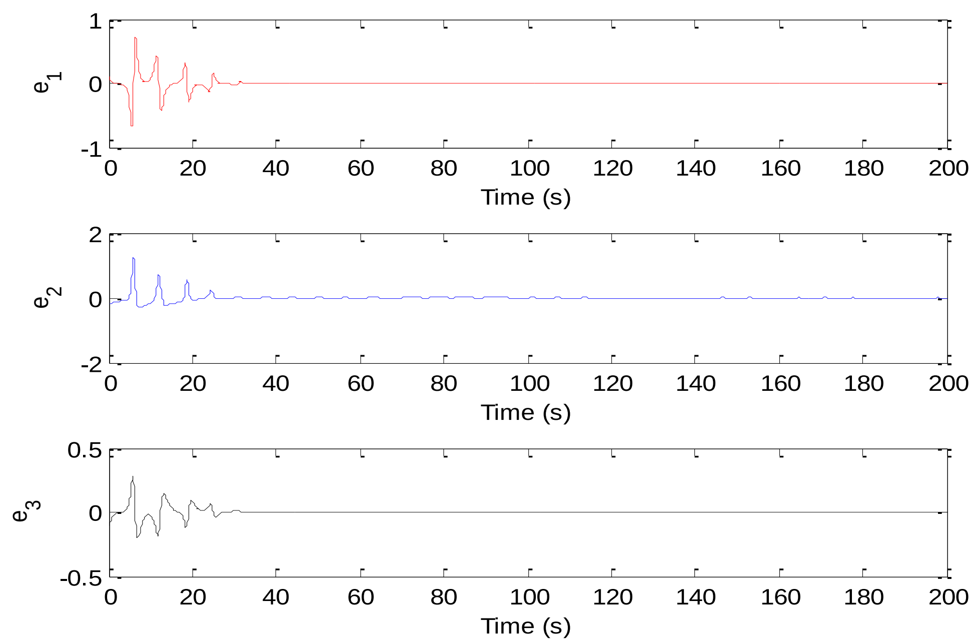

Figure 5. Time response of the error signals of the synchronized finance (3D) systems with the designed controllers.

Deduction from Hyper-chaotic (4D) finance system

Using the fourth order Runge-Kutta routine with the following initial conditions: $(x_1(0), y_1(0)) = (-0.2, -0.1), (x_2(0), y_2(0)) = (0.1, -0.1), (x_3(0), y_3(0)) = (0.2, 0.1), (x_4(0), y_4(0)) = (0.1, -0.2)$ a time-step of 0.001 and fixing the parameter values of $a = 0.9$, $b = 0.2$, $c = 1.2, d = 0.2$, and $e = 0.17$ as in Figure 2, to ensure hyper-chaotic motion of the state variables, we solved the master-slave system (30) with the control matrices as

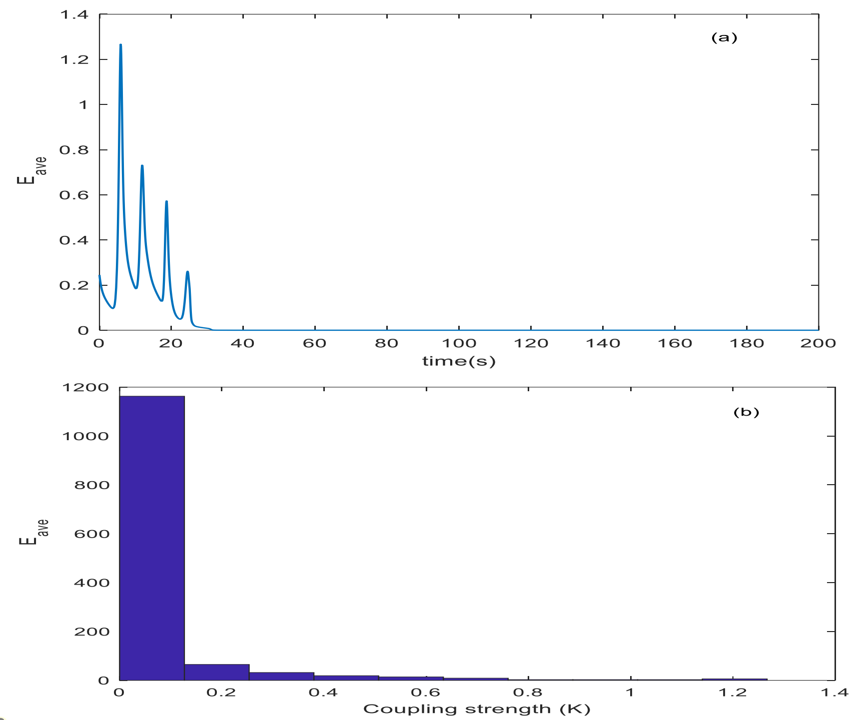

Figure 6. Average error, E_{ave}, as a function of (a) time for the uncoupled systems (b) the coupling strength, k with the following parameters: $= 0.9$, $b = 0.2$, $c = 1.7$.

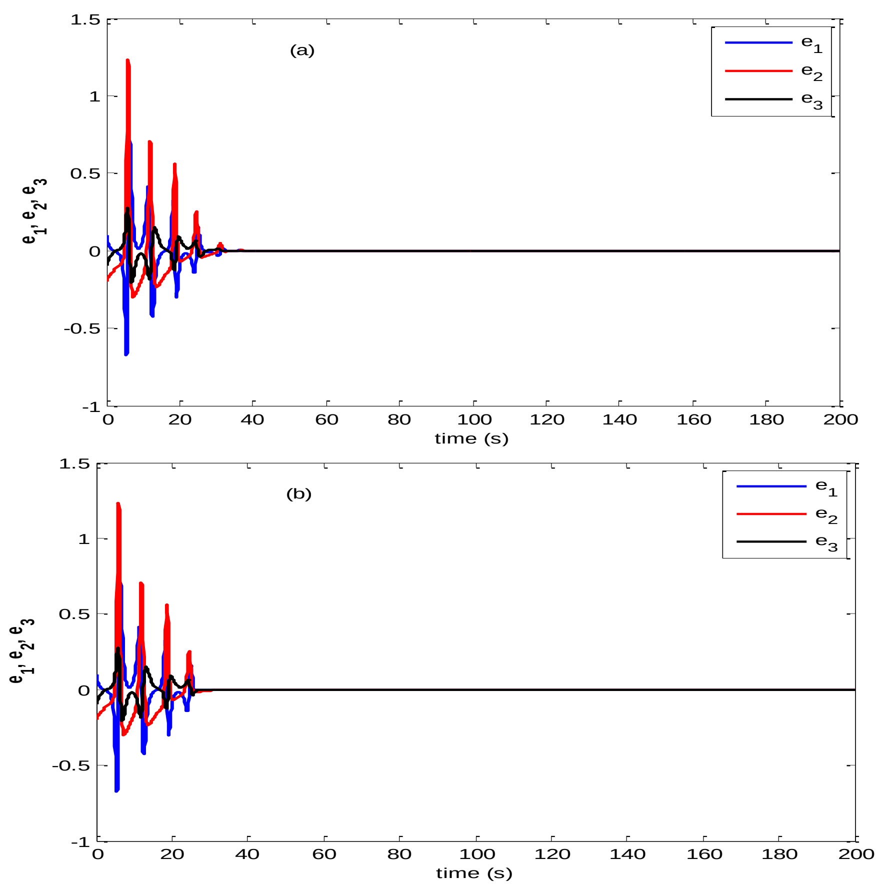

Figure 7. Time response of the global synchronization errors of the coupled hyperchaotic systems with the coupling strength (a) $K = diag\{1.0,1.0,1.0\}$ and (b) $K = diag\{5.0,0.0,0.0\}$.

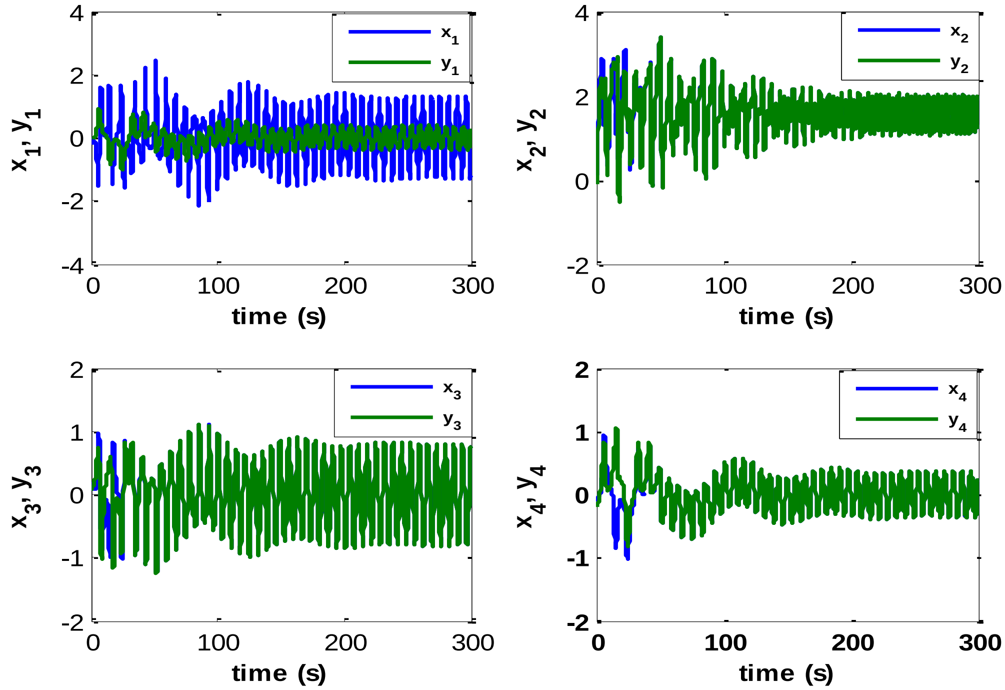

Figure 8. Time series for the synchronized master-slave systems $x = (-0.2, 0.1, 0.2, 0.1)$ and $y = (-0.1, -0.1, 0.1, -0.2)$ respectively with the designed controllers.

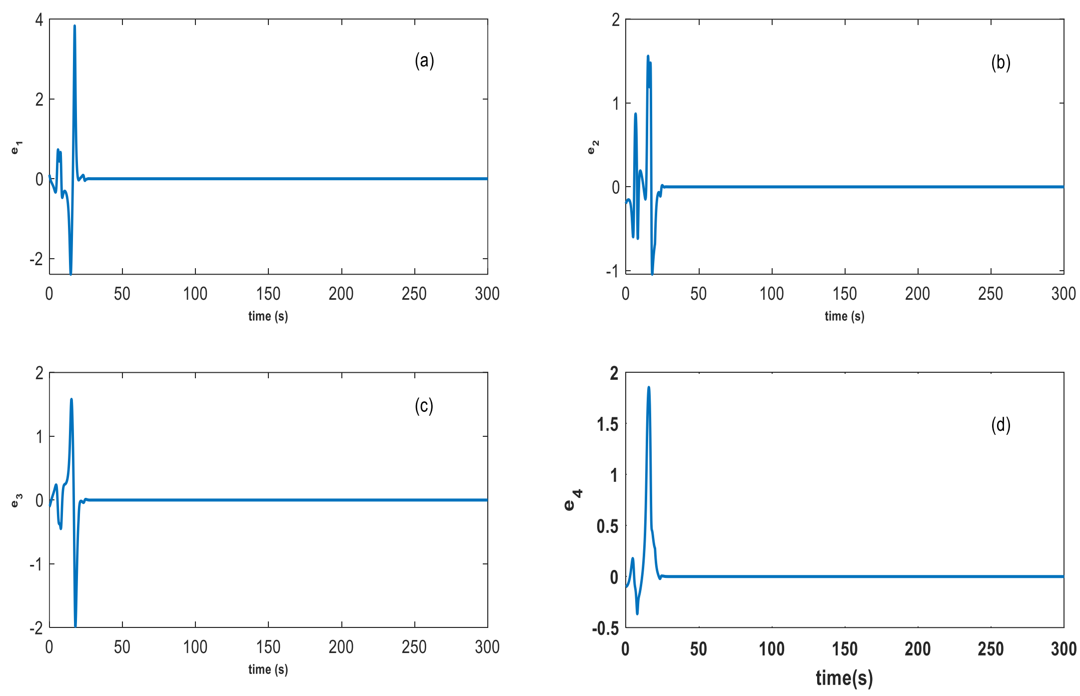

Figure 9. Time response of the error signals of the synchronized finance (4D) systems with the designed controllers.

defined in Eqs. (51) and (54). The simulation results obtained reveal that the trajectory of the master finance system depicted in Figure 2, is bounded, the time series of finance system is shown in Figure 8 and the error dynamics shown in Figure 9 oscillate chaotically with time when the two systems are decoupled. The partial

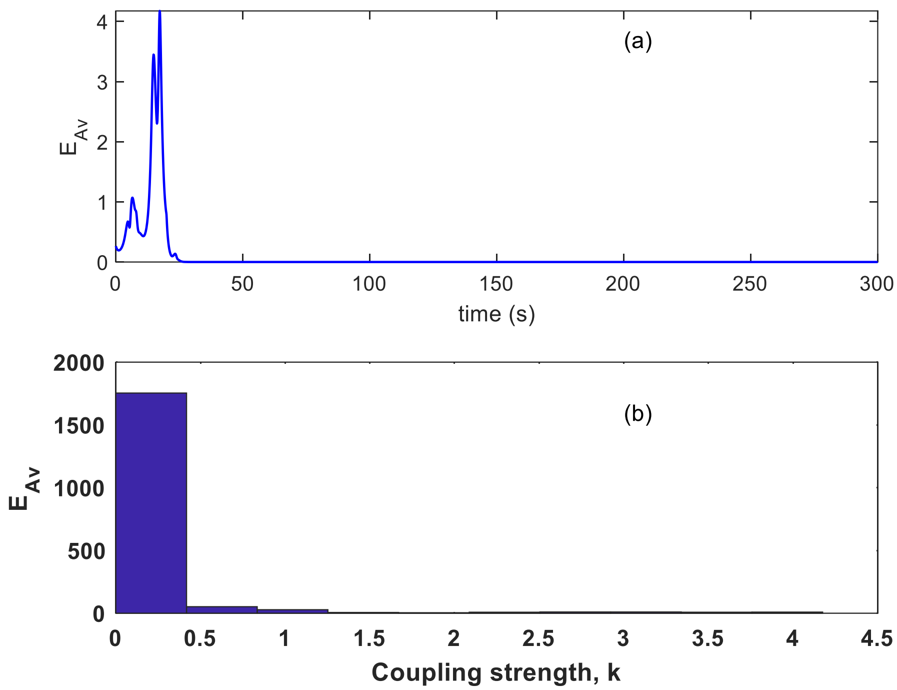

Figure 10. Average error, E_{Av}, as a function of (a) time for the uncoupled systems (b) coupling strength, k with the following parameters: $a = 0.9, b = 0.2, c = 1.2, d = 0.2$ *and* $e = 0.17$ respectively.

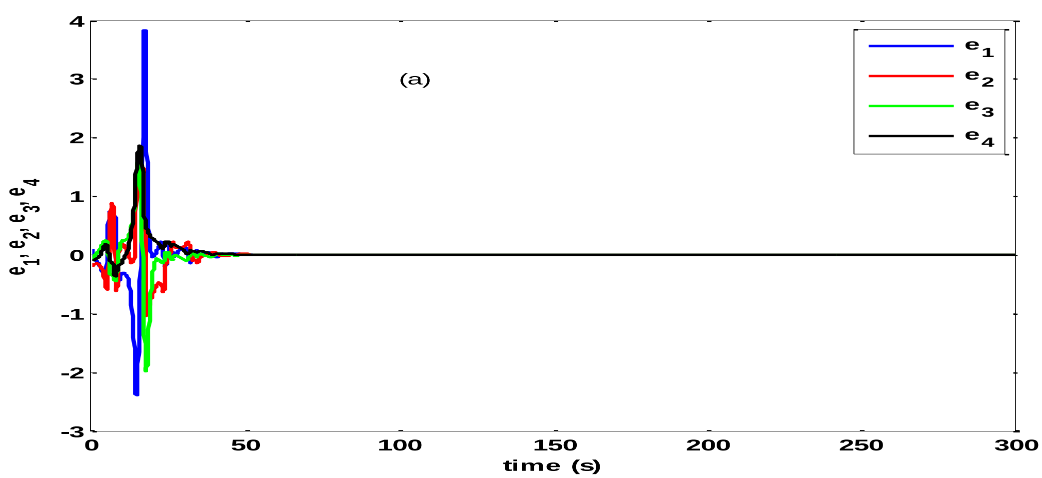

Figure 11. Time response of the global synchronization error of the coupled hyperchaotic systems with the coupling strength (a) $K = diag\{1.0,1.0,1.0,1.0\}$

variables x_1, x_2, x_3 and x_4 of the chaotic attractor satisfy $x_1(\mathrm{t}) = -cx_3(\mathrm{t}) < 1$ for any t ≥ 0.

In Figure 10, we displayed a simulation result of average error, E_{ave}, against coupling, k, and noticed that as k, increases and as full synchronization is approached, $E_{ave} \rightarrow 0$ asymptotically at the threshold coupling, $k_{th} \approx 2.0$. Then for all $k > k_{th}$, E_{ave} = 0 and remains stable as t → ∞ implying that the oscillators are completely synchronized. Interestingly, we noticed that by direct calculations of Eq. (51) for the control matrix, $K = diag\{k, k, k, k\}$, $k > k_{th} = 1.0$. Thus, the obtained criterion is in good agreement with numerical simulation result. Using the criterion defined by Eq. (51), one readily obtains a coupling matrix $K = diag\{1.0,1.0,1.0,1.0\}$ by which the master-slave system (30) achieves chaos synchronization. Figure 11 shows the synchronization for $k = 1.0$. Finally, we depict the simulation results for the second case in which we choose constant control matrix $K = diag\{k, 0,0,0\}$,

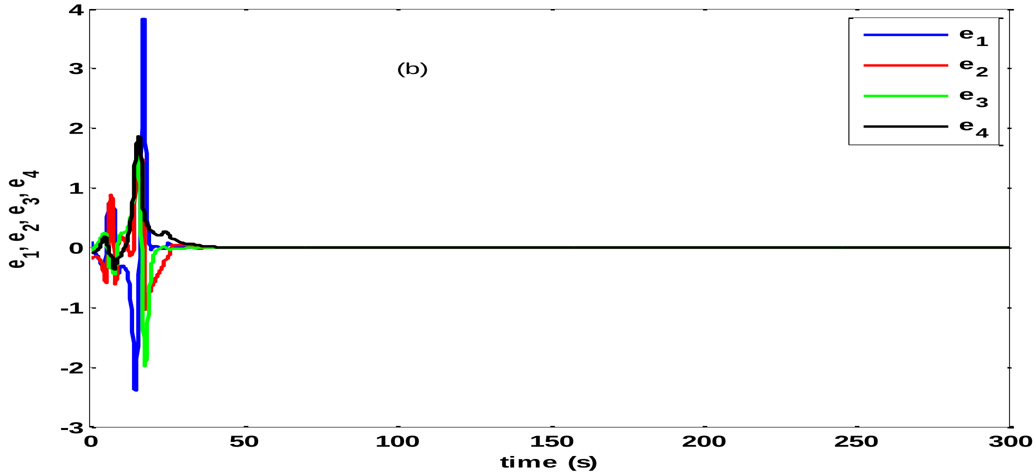

Figure 11 Contd. Time response of the global synchronization error of the coupled hyperchaotic systems with the coupling strength $(b)\ K = diag\{4.2,0,0,0\}$.

such that $k > 4.2$ satisfies the condition in Eq. (54). The simulation results displayed in Figure11 confirmed that complete synchronization is achieved for $k = 4.2 > k_{th}$. Notice that in both cases, the synchronization is already reached at t = 20, showing an excellent transient performance.

Conclusions

In this paper, an analytical method based on Lyapunov stability theory and linear matrix inequality has been utilized to examine the stability of synchronized dynamics and determine the threshold coupling, k_{th}, at which stable synchronization regime could be observed in master-slave parametrically chaotic (3D) and hyperchaotic (4D) finance systems. However, synchronization provides that a low dimensional financial system adapts to the global financial system. Instant variations such as price and interest rate are the main factors of demand and volume changes, and these variations lead to nonlinearity in a system. Therefore, synchronization to the global finance system utilizes some benefits to economic growth on account of obtaining the same interest rate, investment demand and price exponent and also reduces the asymmetrical economic risks. The 3D and 4D finance dynamic behaviour has been thoroughly featured through phase portraits and Lyapunov exponents' diagrams. Based on the criteria utilized, the coupling strength, k_{th} for both 3D and 4D finance systems, by direct calculation was obtained to be approximately 1.0 and 5.0 respectively for both cases of threshold coupling, which also justify the range of values of coupling strength k_{th} according to control theorem. The 3D and 4D systems synchronise perfectly as it can be seen in Figures 6 (b) and 10 (b) respectively. The criteria obtained in this paper are in algebraic form and could be easily employed for designing the feedback control gains that would guarantee complete and stable synchronisation. Finally, numerical simulation results to verify the effectiveness of the obtained criteria were presented.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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