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Full Length Research

Applicability of Varshni potential to predict the mass spectra of heavy mesons and its thermodynamic properties

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ABSTRACT: In this present study, the radial Schrödinger equation is solved analytically with Varshni potential model using the Nikiforov-Uvarov method. The energy equation and corresponding wave function were obtained. The analytical energy expression was used to predict the mass spectra of heavy quarkonia such as charmonium $c\bar{c}$ and bottomonium $b\bar{b}$. Also, the thermodynamic properties such as free energy, mean energy, entropy, and specific heat were obtained. The results obtained with the present model agree excellently with experimental data and the work of other researchers with a maximum error of $0.0055\ GeV$.

Keywords: Heavy quarkonia, Nikiforov-Uvarov method, Schrödinger equation, thermodynamic properties, Varshni potential.

INTRODUCTION

The investigation of thermodynamic properties is significant in various areas of physical and chemical science. This is made conceivable through the solutions of the Schrödinger equation (SE), which contain all the essential data to portray the quantum system under investigation. This area of physics plays a fundamental part in high-energy physics (Florkowski et al., 2004). The thermodynamic properties play a fundamental part in portraying quark-gluon plasma, where the thermodynamic properties of heavy quarks are determined with respect to the strange quark matter (Modarres and Mohamadriejad, 2013). The distribution function needed to determine the thermodynamic properties of any physical system is the partition function (Ikot et al., 2016; Ikot et al., 2018). The solution of the SE with spherically symmetric potentials is of major concern in portraying the mass spectra of heavy quarkonia such as bottomonium and charmonium (Anisiu, 2015). The potential commonly utilized in simulating the interaction for this system is the confining-type potential. For example, a variety of this type of potential is the alleged Cornell potential or Killingbeck potential with two important terms, of Coulomb interaction and confinement of the quarks, respectively (Mocsy, 2009).

Numerous researchers have solved both exact and approximate solutions of SE utilizing diverse insightful methods with potentials to obtained thermodynamic properties of some physical systems (Abu-Shady et al., 2019; Okorie et al., 2018; Oyewumi et al., 2014; Ali et al., 2020; Song et al., 2017; Jia et al., 2017; Ikot et al., 2019; Lutfuoglu, 2018). More so, most researchers have obtained the mass spectra of heavy mesons with the Cornell and extended Cornell potentials (Vega and Flores, 2016; Ciftci and Kisoglu, 2018; Abu-Shady, 2015; Mutuk, 2018; Al-Oun et al., 2015; Inyang et al., 2020c; Mansour and Gamal, 2018; Ibekwe et al., 2020; Abu-Shady and Ikot, 2019). Recently, most researchers have carried out the study of mass spectra with exponential-type potentials. For instance, Inyang et al. (2021a) obtained the mass spectra with Yukawa potential using the Nikiforov-Uvarov (NU) method both in relativistic and non-relativistic regime.

More so, Akpan et al. (2021) obtained the mass spectra of heavy mesons in the non-relativistic model with Huthern-Hellmann potential using the NU method. Furthermore, Inyang et al. (2021i) solved the Klein-Gordon equation analytically via the NU method to obtained the energy eigenvalues and corresponding wavefunction in terms of Laguerre polynomials with the ultra generalized exponential–hyperbolic potential. The results were applied for calculating the mass spectra of heavy mesons such as charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) for different quantum states. Also, Ibekwe et al. (2021) solved the SE with screened Kratzer potential using the series expansion method. The results were used to obtain the mass spectra of heavy mesons. In addition, Inyang et al. (2020a) solved the Schrödinger equation via the series expansion method with class of Yukawa potential to calculate the mass spectra of heavy mesons. The Varshni potential takes the form of Varshni (1957).

$$V(r) = a - \frac{abe^{-\alpha r}}{r} \tag{1}$$

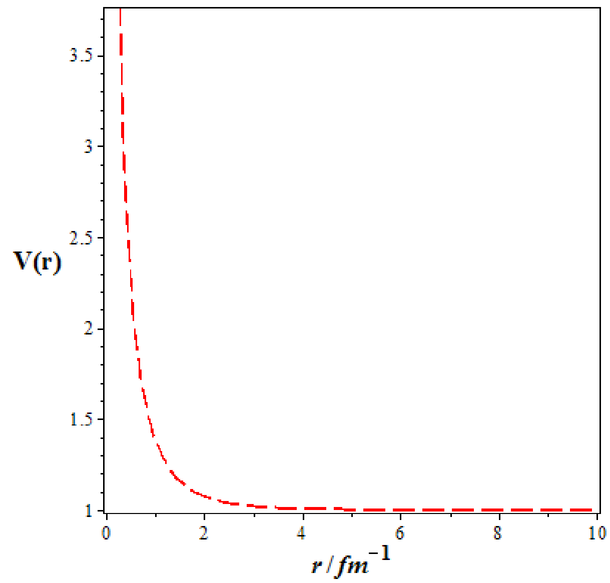

Figure 1. Plots of Varshni potential with r in (fm^{-1}).

where a and b are potential strengths, α is the screening parameter which controls the shape of the potential energy curve as shown in Figure 1 and r the inter-nuclear separation. The Varshni potential is a short range potential with applications cutting across nuclear physics, particle physics and molecular physics (Oluwadare and Oyewumi, 2017). This potential is utilized for the most part to depict bound states interaction of the systems (Lim and Udyavara, 2009). Therefore, this present study aim is in three folds. First, to model the Varshni potential to behave like the Cornell potential, thereafter obtain the solutions using the NU method and finally to calculate the mass spectra and thermodynamic properties of heavy quarks in which the quarks are considered as spineless particles for easiness.

METHODOLOGY

Approximate solutions of the Schrödinger equation with Varshni potential

The SE takes the form (William et al., 2020; Ekpo et al., 2020; Inyang et al., 2020b; Obu et al., 2020):

$$\frac{d^2 \mathrm{U}(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(E_{nl} - V(r)\right) - \frac{l(l+1)}{r^2} \right] \mathrm{U}(r) = 0 \tag{2}$$

where l, μ, r and $\hbar$ are the angular momentum quantum number, the reduced mass for the quarkonium particle, inter-particle distance and reduced plank constant respectively.

Equation (1) is model to interact in the quark-antiquark system by carrying out series expansion of the exponential term and ignoring terms from r^3, this yield

$$\frac{e^{-\alpha r}}{r} = \frac{1}{r} - \alpha + \frac{\alpha^2 \mathrm{r}}{2} - \frac{\alpha^3 r^2}{6} + \ldots \tag{3}$$

We substitute Eq. (3) into Eq. (1) and obtain

$$V(r) = -\frac{B}{r} - Cr + Dr^2 + A \tag{4}$$

Where

$$\left.\begin{aligned} &A = a + ab\alpha, B = ab \\ &C = \frac{ab\alpha^2}{2}, D = \frac{ab\alpha^3}{6} \end{aligned}\right\} \tag{5}$$

Upon substituting Eq. (4) into Eq. (2), we obtain

$$\frac{d^2\,\mathrm{U}(r)}{dr^2} + \left[\frac{2\mu E_{nl}}{\hbar^2} + \frac{2\mu B}{\hbar^2 r} + \frac{2\mu Cr}{\hbar^2} - \frac{2\mu Dr^2}{\hbar^2} - \frac{2\mu A}{\hbar^2} - \frac{l(l+1)}{r^2}\right]\mathrm{U}(r) = 0 \tag{6}$$

Transforming the coordinate from r to y in Eq. (6), we set

$$y = \frac{1}{r} \tag{7}$$

Therefore, the 2nd derivative in Eq. (7) becomes;

$$\frac{d^2\,\mathrm{U(r)}}{dr^2} = 2y^3\frac{dU(\mathrm{y})}{dy} + y^4\frac{d^2\,\mathrm{U(y)}}{dy^2} \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6) we obtain

$$\frac{d^2\,\mathrm{U(y)}}{dy^2} + \frac{2y}{y^2}\frac{dU(\mathrm{y})}{dy} + \frac{1}{y^4}\left[\frac{2\mu E_{nl}}{\hbar^2} + \frac{2\mu By}{\hbar^2} + \frac{2\mu C}{\hbar^2 y} - \frac{2\mu D}{\hbar^2 y^2} - \frac{2\mu A}{\hbar^2} - l(l+1)y^2\right]\mathrm{U(y)} = 0, \tag{9}$$

We suggest the approximation proposal on $\frac{C}{y}$ and $\frac{D}{y^2}$ terms with the assumption that there is a characteristic radius r_0 of the meson. Then the proposal is based on the expansion of $\frac{C}{y}$ and $\frac{D}{y^2}$ in a power series around r_0 ; i.e. around $\delta \equiv \frac{1}{r_0}$, up to the second order. This is similar to Pekeris approximation, which helps to distort the centrifugal term (Abu-Shady, 2016).

Setting $s = y - \delta$ and around $x = 0$ it can be expanded into a series of powers as;

$$\frac{C}{y} = \frac{C}{s+\delta} = \frac{C}{\delta\left(1 + \frac{s}{\delta}\right)} = \frac{C}{\delta}\left(1 + \frac{s}{\delta}\right)^{-1} \tag{10}$$

which yields

$$\frac{C}{y} = C\left(\frac{3}{\delta} - \frac{3y}{\delta^2} + \frac{y^2}{\delta^3}\right) \quad (11)$$

Similarly,

$$\frac{D}{y^2} = D\left(\frac{6}{\delta^2} - \frac{8y}{\delta^3} + \frac{3y^2}{\delta^4}\right) \quad (12)$$

By substituting Eqs. (11) and (12) into Eq. (10), we obtain

$$\frac{d^2\,\mathrm{U(y)}}{dy^2} + \frac{2y}{y^2}\frac{dU(\mathrm{y})}{dy} + \frac{1}{y^4}\left[-\varepsilon + \alpha y - \beta y^2\right]\mathrm{U(y)} = 0 \quad (13)$$

Where

$$\left.\begin{aligned} -\varepsilon &= \left(\frac{2\mu E_{nl}}{\hbar^2} - \frac{2\mu A}{\hbar^2} + \frac{6\mu C}{\hbar^2\delta} - \frac{12\mu D}{\hbar^2\delta^2}\right),\ \alpha = \left(\frac{2\mu B}{\hbar^2} - \frac{2\mu C}{\hbar^2\delta^2} + \frac{16\mu D}{\hbar^2\delta^3}\right) \\ \beta &= \left(\gamma - \frac{2\mu C}{\hbar^2\delta^3} + \frac{6\mu D}{\hbar^2\delta^4}\right),\ \gamma = l(l+1) \end{aligned}\right\} \quad (14)$$

By comparing Eq. (13) and Eq. (A1) we obtain

$$\left.\begin{aligned} &\tilde{\tau}(\mathrm{y}) = 2y,\ \sigma(\mathrm{y}) = y^2 \\ &\tilde{\sigma}(\mathrm{y}) = -\varepsilon + \alpha y - \beta y^2 \\ &\sigma'(\mathrm{y}) = 2y,\ \sigma''(\mathrm{y}) = 2 \end{aligned}\right\} \quad (15)$$

We substitute Eq. (15) into Eq. (A9) and obtain

$$\pi(\mathrm{y}) = \pm\sqrt{\varepsilon - \alpha y + (\beta + k)\, y^2} \quad (16)$$

The value of k is calculated from the function under the square root in Eq. (16), which yield.

$$k = \frac{\alpha^2 - 4\beta\varepsilon}{4\varepsilon} \quad (17)$$

By substituting Eq. (17) into Eq. (16) we have

$$\pi(\mathrm{y}) = \pm\left(\frac{\alpha y}{2\sqrt{\varepsilon}} - \frac{\varepsilon}{\sqrt{\varepsilon}}\right) \quad (18)$$

Taking the negative part of Eq. (18) and differentiating, yields

$$\pi'_{-}(\mathrm{y}) = -\frac{\alpha}{2\sqrt{\varepsilon}} \quad (19)$$

Substituting Eqs. (15) and (18) into Eq. (A7) we have

$$\tau(\mathrm{y}) = 2y - \frac{\alpha y}{\sqrt{\varepsilon}} + \frac{2\varepsilon}{\sqrt{\varepsilon}} \tag{20}$$

Differentiating Eq. (20) gave Eq. (21)

$$\tau'(\mathrm{y}) = 2 - \frac{\alpha}{\sqrt{\varepsilon}} \tag{21}$$

Using Eq. (A10),

$$\lambda = \frac{\alpha^2 - 4\beta\varepsilon}{4\varepsilon} - \frac{\alpha}{2\sqrt{\varepsilon}} \tag{22}$$

And from Eq. (A11), we have

$$\lambda_n = \frac{n\alpha}{\sqrt{\varepsilon}} - n^2 - n \tag{23}$$

Equating Eqs. (22) and (23), the energy eigenvalue of the Varshni potential is obtain as,

$$E_{nl} = a\left(1 + b\alpha\right) - \frac{3ab\alpha^3}{2\delta} + \frac{ab\alpha^3}{\delta^3} - \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu ab}{\hbar^2} - \frac{3\mu ab\alpha^2}{\hbar^2\delta^2} + \frac{16\mu ab\alpha^3}{6\hbar^2\delta^2}}{n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - \frac{ab\mu\alpha^3}{\delta^3\hbar^2} + \frac{16\mu ab\alpha^3}{6\hbar^2\delta^4}}}\right]^2 \tag{24}$$

The wave function is determined by substituting Eqs. (15) and (18) into Eq.(A4) which gives,

$$\frac{d\phi}{\phi} = \left(\frac{\varepsilon}{y^2\sqrt{\varepsilon}} - \frac{\alpha}{2y\sqrt{\varepsilon}}\right)dy \tag{25}$$

By integrating Eq. (25), we obtain

$$\phi(\mathrm{y}) = y^{-\frac{\alpha}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{y\sqrt{\varepsilon}}} \tag{26}$$

Upon substituting Eqs. (15) and (18) into Eq.(A6) and integrating, thereafter simplify to obtain

$$\rho(\mathrm{y}) = y^{-\frac{\alpha}{\sqrt{\varepsilon}}} e^{-\frac{2\varepsilon}{y\sqrt{\varepsilon}}} \tag{27}$$

Substituting Eqs. (15) and (27) into Eq. (A5) gave,

$$\chi_n(\mathrm{y}) = B_n e^{\frac{2\varepsilon}{y\sqrt{\varepsilon}}} y^{\frac{\alpha}{\sqrt{\varepsilon}}} \frac{d^n}{dy^n}\left[e^{-\frac{2\varepsilon}{y\sqrt{\varepsilon}}} y^{2n-\frac{\alpha}{\sqrt{\varepsilon}}} \right] \quad (28)$$

The Rodrigues' formula of the associated Laguerre polynomials is

$$L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{y\sqrt{\varepsilon}}\right) = \frac{1}{n!} e^{\frac{2\varepsilon}{y\sqrt{\varepsilon}}} y^{\frac{\alpha}{\sqrt{\varepsilon}}} \frac{d^n}{dy^n}\left(e^{-\frac{2\varepsilon}{y\sqrt{\varepsilon}}} y^{2n-\frac{\alpha}{\sqrt{\varepsilon}}} \right) \quad (29)$$

Where

$$\frac{1}{n!} = B_n \quad (30)$$

Hence,

$$\chi_n(\mathrm{y}) \equiv L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{y\sqrt{\varepsilon}}\right) \quad (31)$$

Substituting Eqs. (26) and (31) into Eq. (A2), we obtain the wave function of Eq. (6) in terms of Laguerre polynomial as

$$\psi(\mathrm{y}) = N_{nl}\, y^{-\frac{\alpha}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{y\sqrt{\varepsilon}}} L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{y\sqrt{\varepsilon}}\right) \quad (32)$$

Where

N_{nl} is normalization constant, which can be obtain from

$$\int_0^\infty | N_{nl}(r) |^2 dr = 1 \quad (33)$$

Thermodynamic properties of the Schrödinger equation with Varshni potential

Thermodynamic properties of Varshni potential can be obtain from the partition function by using Eq. (24), which reduces to

$$E_{nl} = P_1 - \frac{\hbar^2}{8\mu}\left[\frac{P_2}{(n+\sigma)}\right]^2 \quad (34)$$

where,

$$\sigma = \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{ab\mu\alpha^3}{\hbar^2\delta^3} + \frac{16ab\mu\alpha^3}{6\hbar^2\delta^4}} \quad (35)$$

$$P_1 = a\left(1+b\alpha\right) - \frac{3ab\alpha^3}{2\delta} + \frac{ab\alpha^3}{\delta^3} \tag{36}$$

$$P_2 = \frac{2ab\mu}{\hbar^2} - \frac{3ab\mu\alpha^2}{\hbar^2\delta^2} + \frac{16ab\mu\alpha^3}{6\hbar^2\delta^2} \tag{37}$$

Partition function $Z(\beta)$ **(Abu-Shady et al., 2019)**

The partition function takes the form,

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta E_{nl}} \tag{38}$$

where,

$$\beta = \frac{1}{KT} \tag{39}$$

where K is the Boltzmann constant, T is the absolute temperature, n is the principal quantum number, $n = 0,1,2,3...$ and λ is the maximum or upper bound quantum number.

Substituting Eq. (34) into Eq. (38) we obtain

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta\left(P_1 - \frac{\hbar^2}{8\mu}\left[\frac{P_2}{(n+\sigma)}\right]^2\right)} \tag{40}$$

In the classical limit, at high temperature T , the sum is replaced by an integral,

$$Z(\beta) = \int_0^{\lambda} e^{M_1\beta + \frac{N\beta}{\rho^2}} d\rho \tag{41}$$

where,

$$n + \sigma = \rho \tag{42}$$

$$M_1 = -P_1 \tag{43}$$

$$N = \frac{\hbar^2 P_2^2}{8\mu} \tag{44}$$

Integrating Eq. (41) we obtain the partition function as,

$$Z(\beta)=\frac{1}{2}e^{M_1\beta}\sqrt{N\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}-2\sqrt{N\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta}}-2\sqrt{\pi}\right) \tag{45}$$

And the imaginary error function $erfi(\mathrm{y})$ is defined as follows (Okorie et al., 2018),

$$erfi(\mathrm{y})=\frac{erf\,(iy)}{i}=\frac{2}{\sqrt{\pi}}\int_0^y e^{t^2}dt. \tag{46}$$

Mean energy $U(\beta)$ **(Abu-Shady et al., 2019)**

$$U(\beta)=-\frac{\partial}{\partial\beta}InZ(\beta), \tag{47}$$

Substituting Eq. (45) into Eq. (47) gave,

$$U(\beta)=-\frac{\left[M_1e^{M_1\beta}\sqrt{N\beta}\Delta_1+\frac{\frac{1}{4}e^{M_1\beta}\Delta_1N}{\sqrt{N\beta}}+\frac{1}{2}e^{M_1\beta}\sqrt{N\beta}\Delta_2\right]}{e^{M_1\beta}\sqrt{N\beta}\Delta_1} \tag{48}$$

where,

$$\Delta_1=\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}-2\sqrt{N\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta}}-2\sqrt{\pi} \tag{49}$$

$$\Delta_2=-\frac{\lambda e^{\frac{N\beta}{\lambda^2}}N}{(N\beta)^{\frac{3}{2}}}-\frac{\sqrt{N}\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta^2}}+\frac{N^{\frac{3}{2}}\sqrt{\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{(N\beta)^{\frac{3}{2}}} \tag{50}$$

Free energy $\mathrm{F}(\beta)$

$F(\beta)=-KT\ln Z(\beta)$ (Abu-Shady et al., 2019) (51)

We substitute Eqs. (39) and (45) into Eq.(51) and obtain

$$F(\beta)=-\frac{1}{\beta}\ln\left[\frac{1}{2}e^{M_1\beta}\sqrt{N\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}-2\sqrt{N\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta}}-2\sqrt{\pi}\right)\right] \tag{52}$$

Entropy $\mathrm{S}(\beta)$ **(Abu-Shady et al., 2019)**

$$S(\beta)=K\ln Z(\beta)-K\beta\frac{\partial}{\partial\beta}\ln Z(\beta) \tag{53}$$

We substitute Eqs. (45) and (48) into Eq.(53) and obtain

$$\begin{aligned}S(\beta)=&K\ln\left[\frac{1}{2}e^{M_1\beta}\sqrt{N\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}-2\sqrt{N\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta}}-2\sqrt{\pi}\right)\right]\\&-K\beta\frac{\left(M_1e^{M_1\beta}\sqrt{N\beta}\Delta_1+\frac{\frac{1}{4}e^{M_1\beta}\Delta_1N}{\sqrt{N\beta}}+\frac{1}{2}e^{M_1\beta}\sqrt{N\beta}\Delta_2\right)}{e^{M_1\beta}\sqrt{N\beta}\Delta_1}\end{aligned} \tag{54}$$

Specific heat $\mathrm{C}(\beta)$ **(Abu-Shady et al., 2019)**

$$C(\beta)=\frac{\partial U}{\partial T}=-K\beta^2\frac{\partial U}{\partial\beta} \tag{55}$$

Substituting Eqs. (49) and (51) into Eq. (48) and then substitute into Eq. (55) we obtain

$$C(\beta)=-K\beta^2\left\{\begin{aligned}&\frac{1}{\gamma_1}\left[\begin{aligned}&\left(M_1^2\right)\gamma_1+\frac{M_1e^{M_1\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)-2\sqrt{\pi}\right)N}{\sqrt{N\beta}}-\frac{2M_1e^{M_1\beta}\lambda e^{\frac{N\beta}{\lambda^2}}}{\beta}\\&-\frac{1}{4}\frac{e^{M_1\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)-2\sqrt{\pi}\right)N^2}{\left(N\beta\right)^{\frac{3}{2}}}+\frac{1}{2}\frac{e^{M_1\beta}\lambda e^{\frac{N\beta}{\lambda^2}}}{\beta^2}-\frac{e^{M_1\beta}Ne^{\frac{N\beta}{\lambda^2}}}{3\lambda}\end{aligned}\right]\\&-\frac{1}{\gamma_1}\left[M_1\gamma_1+\frac{\gamma_2}{2}-\frac{e^{M_1\beta}\lambda e^{\frac{N\beta}{\lambda^2}}}{\beta}\right]\\&-\frac{1}{\gamma_3}\left[\begin{aligned}&\left[\frac{1}{2}\left(M_1e^{M_1\beta}\right)\sqrt{N\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)-2\sqrt{\pi}\right)+\frac{\gamma_2}{4}-\frac{1}{2}\frac{e^{M_1\beta}}{\beta}\right]N\\&+\left[M_1\gamma_1+\frac{\gamma_2}{2}-\frac{e^{M_1\beta}+\lambda e^{\frac{N\beta}{\lambda^2}}}{\beta}\right]\lambda e^{\frac{N\beta}{\lambda^2}}\end{aligned}\right]\end{aligned}\right\} \tag{56}$$

Where

$$\gamma_1 = e^{M_1\beta}\sqrt{N\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}} - 2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right) - 2\sqrt{\pi}\right) \tag{57}$$

$$\gamma_2 = \frac{e^{M_1\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}} - 2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right) - 2\sqrt{\pi}\right)N}{\sqrt{N\beta}} \tag{58}$$

$$\gamma_3 = e^{M_1\beta}N\beta^2\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}} - 2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right) - 2\sqrt{\pi}\right)^2 \tag{59}$$

Also, the mass spectra of the heavy quarkonia such as charmonium and bottomonium using the relation (Inyang et al., 2021b; Inyang et al., 2021c) was calculated.

$$M = 2m + E_{nl} \tag{60}$$

where m is quarkonium bare mass, and E_{nl} is energy eigenvalues. By substituting Eq. (24) into Eq. (60) we obtain the mass spectra for Varshni potential as:

$$M = 2m + a\left(1+b\alpha\right) - \frac{3ab\alpha^3}{2\delta} + \frac{ab\alpha^3}{\delta^3} - \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu ab}{\hbar^2} - \frac{3\mu ab\alpha^2}{\hbar^2\delta^2} + \frac{16\mu ab\alpha^3}{6\hbar^2\delta^2}}{n + \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{ab\mu\alpha^3}{\delta^3\hbar^2} + \frac{16\mu ab\alpha^3}{6\hbar^2\delta^4}}}\right]^2 \tag{61}$$

RESULTS AND DISCUSSION

In the calculation, the heavy quark masses were chosen as 4.680 GeV and 1.488 GeV, for bottomonium and charmonium respectively (Barnett et al., 2012). The corresponding reduced mass are $\mu_b =$ 2.340 GeV and $\mu_c =$ 0.744 GeV. The values of the potential parameters of Eq. (61) for the different mesons are fixed by fitting the experimental data. Experimental data is taken from Tanabashi et al. (2018).

The mass spectra of heavy mesons was calculated in comparison with experimental data and recent theoretical works (Inyang et al., 2021a; Inyang et al., 2021i) in which they used different models as presented in Tables 1 and 2. It was noted that there was an improvement with the present model. In order to test for the accuracy of the predicted results determined analytically, a Chi square function was used to determine the error between the experimental data and theoretical predicted values. The maximum error in comparison with the experimental data is found to be 0.0055 GeV.

The thermodynamic properties were obtained by first obtaining the partition function. Figure 2 show that the partition function $Z(\beta)$ decreases exponentially with increasing temperature β for different values of maximum quantum number λ, then later increases with increasing temperature which is the same as reported by

Table 1. Mass spectra of charmonium in (GeV) $(m_c = 1.488$ GeV, $\mu = 0.744$ GeV $\alpha = -0.976$, $\delta = 1.700$ GeV, $\hbar = 1, a = -48.049$ GeV and $b = 3.020$.

State	Present work	Abu-Shady (2016)	Inyang et al. (2021a)	Experiment (Tanabashi et al., 2018)
1S	3.096	3.096	3.096	3.096
2S	3.686	3.686	3.686	3.686
1P	3.524	3.255	3.527	3.525
2P	3.768	3.779	3.687	3.773
3S	4.040	4.040	4.040	4.040
4S	4.264	4.269	4.360	4.263
1D	3.683	3.504	3.098	3.770
2D	3.989	-	3.976	4.159
1F	3.862	-	4.162	-

Table 2. Mass spectra of bottomonium in (GeV) $(m_b = 4.680$ GeV, $\mu = 2.340$ GeV, $\alpha = -0.952$, $\delta = 1.70$ GeV, $\hbar = 1, a = -14.352$ GeV and $b = 3.084$ $GeV)$.

State	Present work	Abu-Shady (2016)	Inyang et al. (2021a)	Experiment (Tanabashi et al., 2018)
1S	9.460	9.460	9.460	9.460
2S	10.023	10.023	10.023	10.023
1P	9.789	9.619	9.661	9.899
2P	10.243	10.114	10.238	10.260
3S	10.355	10.355	10.355	10.355
4S	10.579	10.567	10.567	10.580
1D	9.998	9.864	9.943	10.164
2D	10.307	-	10.306	-
1F	10.223	-	10.209	-

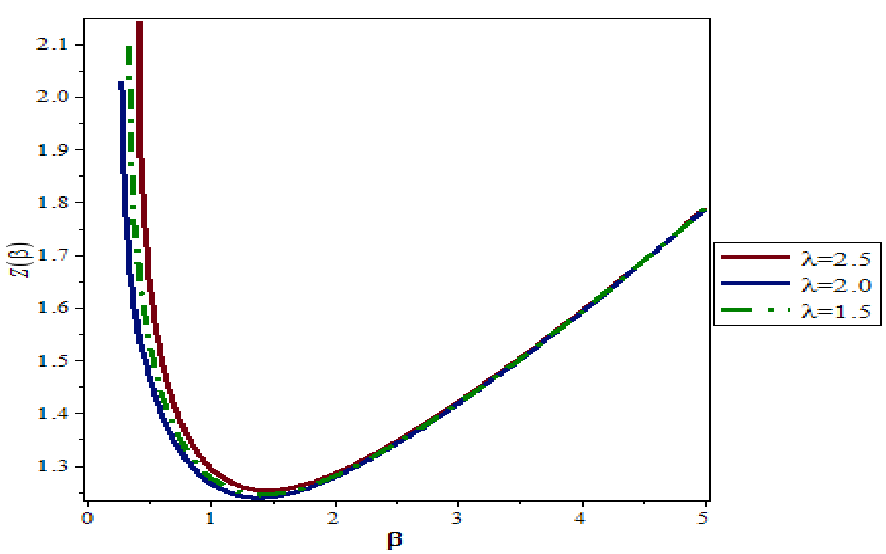

Figure 2. Variation of the partition function $Z(\beta)$ versus temperature (β) for different values of maximum quantum number (λ).

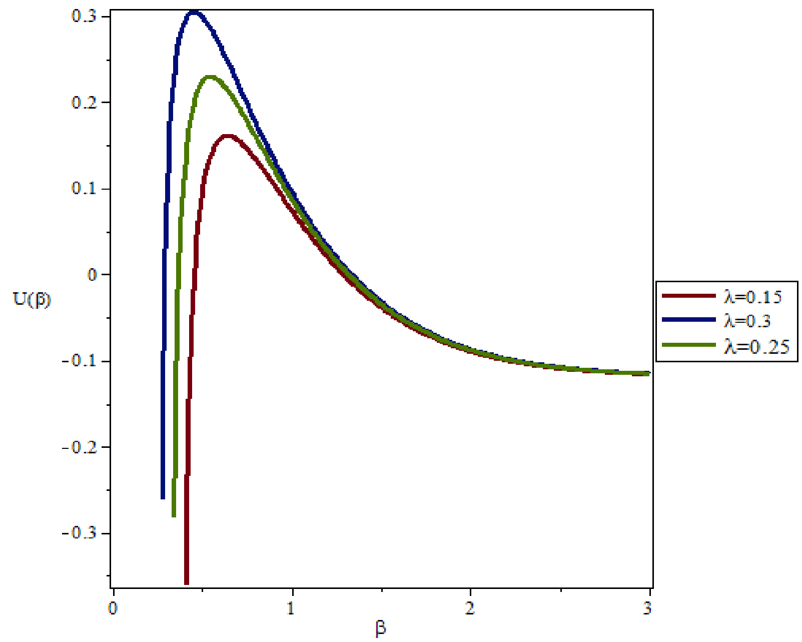

Figure 3. Variation of the mean energy $U(\beta)$ versus temperature (β) for different values of maximum quantum number (λ).

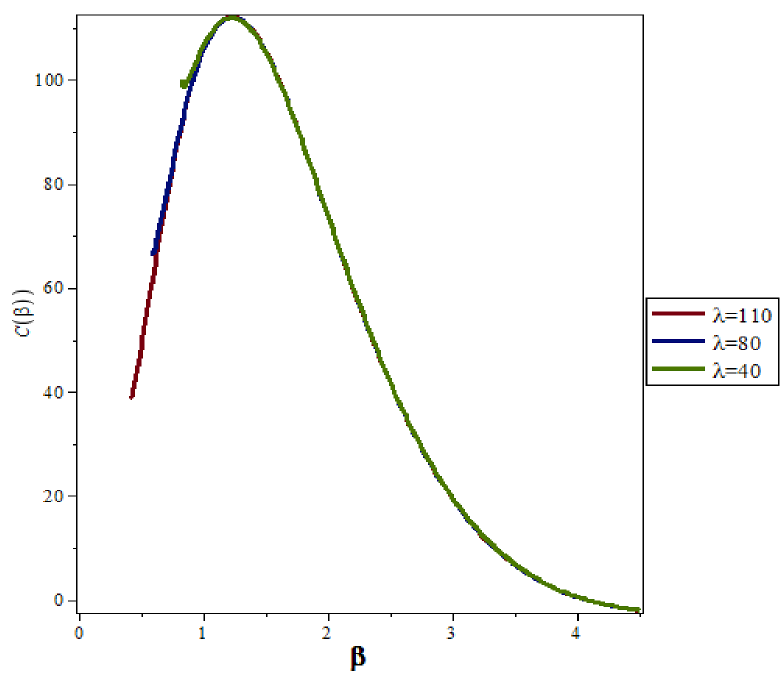

Figure 4. Variation of the specific heat $C(\beta)$ versus temperature (β) for different values of maximum quantum number (λ).

Abu-Shady et al. (2019), the authors studied the thermodynamic properties of a heavy quarkonium system using the NU method. The plot of mean energy $U(\beta)$ with different values of β and λ is shown in Figure 3. It depicts a monotonic increase and then decreases with increase values of β and λ. Figure 4 show the plot of specific heat $C(\beta)$. It is seen to increase monotonically as β increases and then decreases as β and λ increases

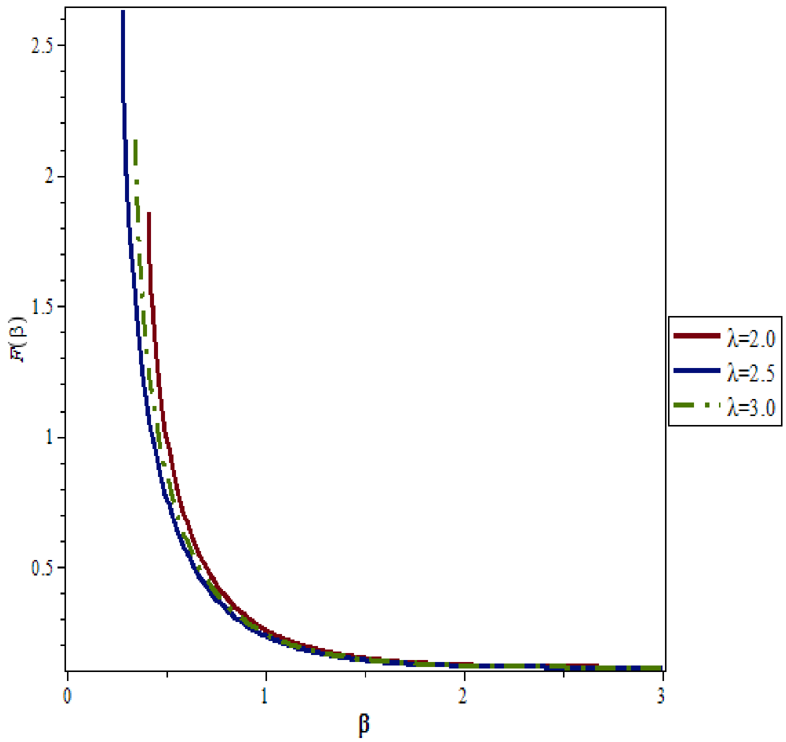

Figure 5. Variation of the free energy $F(\beta)$ versus temperature (β) for different values of maximum quantum number (λ).

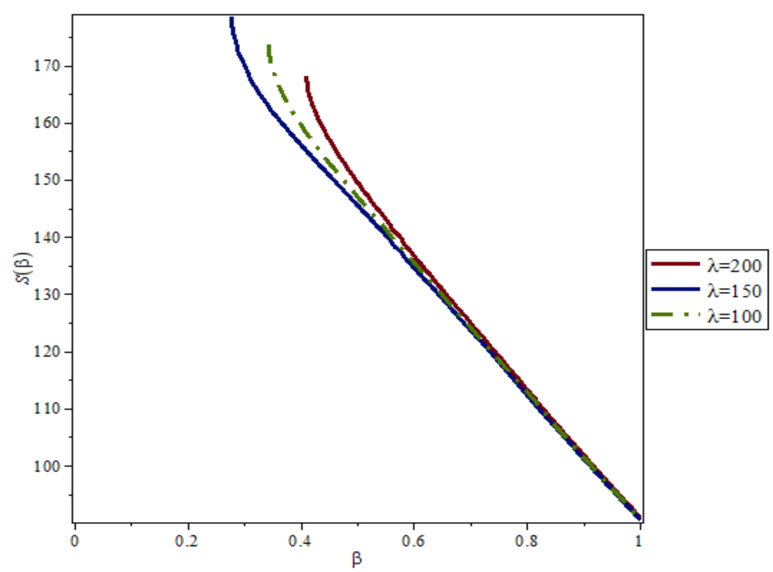

Figure 6. Variation of the entropy $S(\beta)$ versus temperature (β) for different values of maximum quantum number.

with each plot converging. The free energy $F(\beta)$ is plotted as shown in Figure 5. The free energy is seen to decrease exponentially as β and λ increases and converges at a point close to zero. The plot of entropy $S(\beta)$ as a function of temperature β and maximum quantum number λ is shown in Figure 6. It was noted that the entropy decreases with increasing β . This finding is in agreement with Okorie et al. (2018) in which the entropy increases with increasing temperature for the system.

Conclusion

In this work, the Varshni potential was modeled to interact in the quarkonium system. The Schrödinger equation was broken down with the modeled potential using the NU method for energy eigenvalues and the corresponding eigenfunction in terms of Laguerre polynomials. The present results were applied to compute mass spectra of heavy quarkonia such as charmonium and bottomonium for different quantum states. The results agree with experimental data and shows an improvement from recent theoretical studies with a maximum error of $0.0055\ GeV$. The thermodynamic properties such as free energy, mean energy, entropy, and specific heat and their plots were also obtained.

COMPETING INTERESTS

The authors declare that they have no competing interests.

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APPENDIX

Review of Nikiforov-Uvarov (NU) methods

The NU method (Nikiforov and Uvarov, 1988; Ntibi et al., 2020; Okoi et al., 2020; Edet and Okoi, 2019; Inyang et al., 2021d; Inyang et al., 2021e; Inyang et al., 2021f; Inyang et al., 2021g; Inyang et al., 2021h; Inyang et al., 2021i; Nwabuzor et al., 2021) is used to solve the second-order differential equation which takes the following form:

$$\psi''(y)+\frac{\tilde{\tau}(y)}{\sigma(y)}\psi'(s)+\frac{\tilde{\sigma}(y)}{\sigma^2(y)}\psi(y)=0 \qquad \text{(A1)}$$

where $\tilde{\sigma}(y)$ and $\sigma(y)$ are polynomials of maximum second degree and $\tilde{\tau}(y)$ is a polynomial of maximum first degree. The exact solution of Eq. (2) takes the form

$$\psi(y)=\phi(y)\chi(y) \qquad \text{(A2)}$$

Substituting Eq. (3) into Eq. (2), we obtain

$$\sigma(y)\chi''(y)+\tau(y)\chi'(y)+\lambda\chi(y)=0 \qquad \text{(A3)}$$

Where the function $\phi(y)$ satisfies the following relation

$$\frac{\phi'(y)}{\phi(y)}=\frac{\pi(y)}{\sigma(y)} \qquad \text{(A4)}$$

And $\chi(y)$ is a hypergeometric-type function, whose polynomial solutions are obtained from the Rodrigues relation

$$\chi_n(y)=\frac{B_n}{\rho(y)}\frac{d^n}{dy^n}\left[\sigma^n(y)\rho(y)\right] \qquad \text{(A5)}$$

where B_n is the normalization constant and $\rho(y)$ the weight function which satisfies the condition below;

$$\frac{d}{dy}\left(\sigma(y)\rho(y)\right)=\tau(y)\rho(y) \qquad \text{(A6)}$$

Where also

$$\tau(y)=\tilde{\tau}(y)+2\pi(y) \qquad \text{(A7)}$$

For bound solutions, it is required that

$$\frac{d\tau(y)}{dy}<0 \qquad \text{(A8)}$$

We can then obtain the eigenfunction and eigenvalues using the definition of the following function $\pi\left(y\right)$ and parameter λ, given as:

$$\pi\left(y\right)=\frac{\sigma'\left(y\right)-\tilde{\tau}\left(y\right)}{2}\pm\sqrt{\left(\frac{\sigma'\left(y\right)-\tilde{\tau}\left(y\right)}{2}\right)^{2}-\tilde{\sigma}\left(y\right)+k\sigma\left(y\right)} \tag{A10}$$

And

$$\lambda=k+\pi'\left(y\right) \tag{A11}$$

The value of k can be calculated if the function under the square root in Eq. (10) is the square of a polynomial. This is possible if its discriminate is equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda_{n}+n\tau^{'}(\mathrm{y})+\frac{n(n-1)}{2}\sigma^{''}(\mathrm{y})=0 \tag{A12}$$