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Convexity properties and inequalities regarding the q-Gamma function

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ABSTRACT: This paper is motivated by the work of K. Nantomah in 2017. In the paper, some convexity properties and some inequalities for the (p, k)-analogue of the Gamma function, $\Gamma_{p,k}(x)$ were given. The method engaged in displaying the result makes use of the (p, k)-analogue of the Gamma function. In addition, H$\ddot{o}l$der's integral inequality, Young's inequality and some basic definitions of a convex function were used. As a result, the (p, k)-generalization of some known outcomes concerning the classical gamma function was specified. The fundamental objective of this paper is to ascertain some convexity properties and some inequalities regarding the q-analogue of the Gamma function, $\Gamma_q(x)$. First, utilizing similar techniques as K. Nantomah, the convexity property of the q-Gamma function was demonstrated. Next, exploiting Young's inequality, some inequalities regarding the q-Gamma function were substantiated. At the end, the q-analogue of some accepted results concerning the classical Gamma function was proven.

Keywords: Convex function, Gamma function, q-analogue of exponential function, q-analogue of gamma function, psi function, logarithmically convex.

INTRODUCTION AND PRELIMINARIES

Let $\Gamma(x)$ be the renowned established Gamma function defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t}\ t^{x-1}\,dt$$

The psi function $\Psi(x)$ function, otherwise known as digamma function, is described as the logarithmic derivative of the gamma function (Abramowitz and Stegun, 1965). That is,

$$\Psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

The q- Gamma function (also recognized as the q-analogue of the Gamma function) is expressed for q $\in (0,1)$ $and\ x > 0$ as

$$\Gamma_q(x) = \int_0^{[\infty]_q} E^{(-qt)}\ t^{x-1} d_q t \qquad (1)$$

Where $E_q^x = \sum_{n=0}^{\infty} \binom{n}{2} = \frac{x^n}{[n]_q!}$ is a q- analogue of exponential function.

When q = 1 in the definition yields to the meaning of the classical Gamma function.

The function $\Gamma_q(x)$ exhibits the following properties.

$\Gamma_q(x+1) = [x]_q \Gamma_q(x)$

$\Gamma_q(1) = 1$

$\Gamma_q(n+1) = [n]_q\,! \qquad n = 0,1,2,3, \ldots.$

Where $[x]_q = \frac{1-q^x}{1-q}$. See Njionou Sadjang (2017).

Let $\Psi_q(x)$ be the q-analogue of the psi function likewise expressed for $x > 0$ as

$$\Psi_q(x) = \frac{d}{dx} \ln\left(\Gamma_q(x)\right) = \frac{\Gamma'_q(x)}{\Gamma_q(x)}.$$

In literature, it is well-known that this function is increasing for $x > 0$. See Mansour and Shaban (2008).

FUNDAMENTAL DEFINITIONS AND CONCEPTS

Definition 1. A function $\mathrm{f}: (\mathrm{a}, \mathrm{b}) \rightarrow \mathrm{R}^+$ is said to be convex if

$\mathrm{f}(\alpha \mathrm{x} + \beta \mathrm{y}) \leq \alpha \mathrm{f}(\mathrm{x}) + \beta \mathrm{f}(\mathrm{y})$

For all $\mathrm{x}, \mathrm{y} \in (\mathrm{a}, \mathrm{b})$ where α, β > 0 such that α + β = 1.

Definition 2. A function $\mathrm{f}: (\mathrm{a}, \mathrm{b}) \rightarrow \mathrm{R}^+$ is said to be logarithmically convex if the inequality

$\log \mathrm{f}(\alpha \mathrm{x} + \beta \mathrm{y}) \leq \alpha \log \mathrm{f}(\mathrm{x}) + \beta \log \mathrm{f}(\mathrm{y})$

Or equivalently

$\mathrm{f}(\alpha \mathrm{x} + \beta \mathrm{y}) \leq \left(\mathrm{f}(\mathrm{x})\right)^{\alpha} (\mathrm{f}(\mathrm{y}))^{\beta}$

Holds for all $\mathrm{x}, \mathrm{y} \in (\mathrm{a}, \mathrm{b})$ and α, β > 0 such that α + β = 1 (Merkle, 2008).

A convex function displays the following basic properties:

If $h(x)$ is convex, then $g(x) = m.h(x)$ is also convex for any positive constant multiplier m.

If $h(x)$ and $g(x)$ are convex, then their addition $p(x) = h(x) + g(x)$ is convex.

If $h(x)$ is convex, then $g(x) = h(qx + r)$ is also convex for any constants $q, r \in R$. But the interval of convexity will change.

It is recognized in literature that these properties can be used to combine basic convex functions to build new complex convex functions (Loh, 2013).

The ultimate objective of this paper is to establish some convexity properties and some inequalities regarding the q-analogue of the Gamma function, $\Gamma_q(x)$ using a similar technique as Nantomah (2017).

RESULT and DISCUSSION

Convexity properties Involving the q-Gamma function

Theorem 1

The function $\Gamma_q(x)$ is logarithmically convex.

Proof. Let *x, y* > 0 and *α, β* > 0 such that *α* + *β* = 1. Then, by the definition of q-Gammer function (1) and by the H$\ddot{o}lder's$ inequality for the Jackson's q-integral, we obtain

$$\begin{aligned}\Gamma_q(\alpha x+\beta y) &= \int_0^{[\infty]_q} E^{(-qt)}\ t^{\alpha x+\beta y-1}\, d_q t \\ &= \int_0^{[\infty]_q} E^{(-qt)(\alpha+\beta)}\ t^{\alpha(x-1)}\ t^{\beta(y-1)}\, d_q t \\ &= \int_0^{[\infty]_q} E^{(-qt)\alpha}\ t^{\alpha(x-1)}\ E^{(-qt)\beta}\ t^{\beta(y-1)}\, d_q t \\ &\le \left(\int_0^{[\infty]_q} E^{(-qt)}\ t^{(x-1)}\, d_q t\right)^{\alpha} \left(\int_0^{[\infty]_q} E^{(-qt)}\ t^{(y-1)}\, d_q t\right)^{\beta} \\ &= \left(\Gamma_q(x)\right)^{\alpha}\left(\Gamma_q(y)\right)^{\beta}.\end{aligned}$$

As required.

Remark 1. It is prominent in literature that every logarithmically convex function is also convex (Niculescu and Persson, 2009).

Consequently, the function $\Gamma_q(x)$ is convex.

Theorem 2

Let $q \in (0,1)$. Then the inequality

$$\Gamma_q\left(\frac{x+y}{2}\right) \le \sqrt{\Gamma_q(x)\Gamma_q(y)}$$

holds for $x, y > 0$.

Proof. This follows right from theorem 1 by letting $\alpha, \beta = \frac{1}{2}$.

Theorem 3

Let q ∈ (0,1) and a > 0. Then the function $A(x) = a^x\,\Gamma_q(x)$ is convex on $(0,\infty)$.

Proof. Remember that $\Gamma_q(x)$ is logarithmically convex.

Thus, $\Gamma_q(\alpha x+\beta y) \le \left(\Gamma_q(x)\right)^{\alpha}\left(\Gamma_q(y)\right)^{\beta}$ (2)

For $x, y > 0$ and α, β > 0 such that α + β = 1. Then,

$$A(\alpha x+\beta y) = a^{\alpha x+\beta y}\,\Gamma_q(\alpha x+\beta y) \le a^{\alpha x+\beta y}\left(\Gamma_q(x)\right)^{\alpha}\left(\Gamma_q(y)\right)^{\beta} \quad (3)$$

Also recall from the Young's inequality that

$$u^{\alpha}v^{\beta} \le \alpha u+\beta v \quad (4)$$

For $u, v > 0$ and *α, β* > 0 such that *α* + *β* = 1. Let $u = a^x\Gamma_q(x)$ *and* $v = a^y\Gamma_q(y)$.

Then (4) becomes

$$a^{\alpha x+\beta y}\left(\Gamma_q(x)\right)^{\alpha}\left(\Gamma_q(y)\right)^{\beta} \leq \alpha a^x\Gamma_q(x) + \beta a^y\Gamma_q(y) = \alpha A(x) + \beta A(y) \quad (5)$$

Combining (3) and (5) yields $A(\alpha x + \beta y) \leq \alpha A(x) + \beta A(y)$ which completes the proof.

Theorem 4

Let q ∈ (0,1). Then the function $B(x) = e^x\Gamma_q(x)$ is convex on $(0,\infty)$.

Proof. Recollect that $\Gamma_q(x)$ is logarithmically convex. Thus,

$$\Gamma_q(\alpha x + \beta y) \leq \left(\Gamma_q(x)\right)^{\alpha}\left(\Gamma_q(y)\right)^{\beta} \quad (6)$$

For *x, y* > 0 and *α, β* > 0 such that *α* + *β* = 1. Then,

$$B(\alpha x + \beta y) = e^{\alpha x+\beta y}\,\Gamma_q(\alpha x + \beta y) \leq e^{\alpha x+\beta y}\left(\Gamma_q(x)\right)^{\alpha}\left(\Gamma_q(y)\right)^{\beta} \quad (7)$$

Similarly recall from the Young's inequality that

$$u^{\alpha}v^{\beta} \leq \alpha u + \beta v \quad (8)$$

For u, v > 0 and α, β > 0 such that α + β = 1. Let $u = e^x\Gamma_q(x)$ and $v = e^y\Gamma_q(y)$.

Then (8) becomes

$$e^{\alpha x+\beta y}\left(\Gamma_q(x)\right)^{\alpha}\left(\Gamma_q(y)\right)^{\beta} \leq \alpha e^x\Gamma_q(x) + \beta e^y\Gamma_q(y) = \alpha B(x) + \beta B(y) \quad (9)$$

Combining (7) and (9) yields $B(\alpha x + \beta y) \leq \alpha B(x) + \beta B(y)$ which concludes the proof of theorem (3)

Theorem 5

Let q ∈ (0,1). Then the function $C(x) = e\Gamma_q(x)$ where e is the Euler's number is convex on $(0,\infty)$.

Proof. It is famous in literature that the function $g(x) = cf(x)$ is convex for any

Positive constant multiplier c if $f(x)$ is convex (Loh, 2013). And since e is a positive constant multiplier and $\Gamma_q(x)$ is convex. Then the function $C(x) = e\Gamma_q(x)$ is convex as required.

Lemma 1. Let $f\colon(0,\infty) \to (0,\infty)$ be a differentiable and logarithmically convex function (Neuman, 2006). Then the function

$g(x) = \frac{(f(x))^{\lambda}}{f(x)}$, $\lambda \geq 1$ Is decreasing on it domain.

Theorem 6

Let q ∈ (0, 1) and λ ≥ 1. Then the inequality

$$\frac{\left(\Gamma_q(1+y)\right)^{\lambda}}{\Gamma_q(1+\lambda y)} \leq \frac{\left(\Gamma_q(1+x)\right)^{\lambda}}{\Gamma_q(1+\lambda x)} \leq 1 \quad (10)$$

Holds true for $0 \leq x \leq y$.

Proof. Let $f(x) = \Gamma_q(1+x)$ for x ≥ 0. It is known that the function f is logarithmically convex. By Lemma (1), $g(x) = \frac{(\Gamma_q(1+x))^\lambda}{\Gamma_q(1+\lambda x)}$ is decreasing and for 0 ≤ x ≤ y, we have $g(y) \leq g(x) \leq g(0)$ yielding the results.

Remark 2. Inequality (10) provides a generalization of Neuman (2006) [Theorem 2.4].

Theorem 7

Let $q \in (0,1), y > 0$ and $\alpha \geq 1$. Then the inequality

$$\frac{\Gamma_q(1+y)^\alpha}{\Gamma_q(\alpha+y)} \leq \frac{\Gamma_q(x+y)^\alpha}{\Gamma_q(\alpha x+y)} \leq 1 \qquad (11)$$

is acceptable for $x \in [0,1]$.

Proof. Express H by $H(x) = \frac{\Gamma_q(x+y)^\alpha}{\Gamma_{q(\alpha x+y)}}$ for $q \in (0,1), y > 0$ and $\alpha \geq 1$. Let $\mu(x) = \ln H(x)$.

Then

$$\mu'(x) = \alpha\frac{\Gamma'_q(x+y)}{\Gamma_q(x+y)} - \alpha\frac{\Gamma'_q(\alpha x+y)}{\Gamma_q(\alpha x+y)}$$

$$= \alpha\left[\Psi_q(x+y) - \Psi_q(\alpha x+y)\right]$$

$$\leq 0$$

Since $\Psi_q(x)$ is increasing (Mansour and Shaban, 2008) for $x > 0$. Hence $H(x)$ is decreasing on $[0,\infty)$. Then for $x \in [0,1]$, we obtain $H(1) \leq H(x) \leq H(0)$ yielding the desire result (11).

Remark 3. As $q \to 1$ in (11), we recover the result of Sandor (2005) [Theorem 2.3], as a special case.

Conclusion

From the above results, some inequalities and convexity properties regarding the q-gamma function have been established. From the proven result, some known results in the literature were claimed.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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