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Full Length Research

Some inequalities bounding certain ratios of the q-gamma function

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ABSTRACT: In this short paper, we established q- analogues of some well-known inequalities for the classical Gamma function and some inequalities bounding the ratio $\Gamma_q(x)/\Gamma_q(y)$, where $\Gamma_q(.)$ is the q-analogue of the Gamma function. The method employed in presenting the results made use of strict logarithmic convexity of a function and classical mean value theorem involving the q-analogue of the psi function. At the end, the proven result provided the q-analogue of some known result concerning the classical Gamma function.

Keywords: Mean value theorem, psi function, q-analogue of the psi function, q-gamma functions, strict logarithmic convexity.

INTRODUCTION AND PRELIMINARIES

The well-known classical Euler's Gamma function, $\Gamma(x)$ is commonly expressed as

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \qquad x > 0$$

The psi function $\psi(x)$, then known as digamma function, is defined as the logarithmic derivative of the gamma function (Abramowitz and Stegun, 1965). That is,

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

It is public information that special functions $\Gamma(x)$ and $\psi(x)$ are essential and significant and have much broad uses in mathematical disciplines.

The q-analogues of Γ and ψ are described by Andrews *et al*. (1999. pp. 493-496) for $q \in (0,1)$ and $x > 0$ by

$$\Gamma_q(x) = \int_0^{[\infty]_q} E^{(-qt)}\, t^{x-1} d_q t$$

And

$$\psi(x) = \frac{d}{dx}\ln\left(\Gamma_q(x)\right) = \frac{\Gamma'_q(x)}{\Gamma_q(x)}$$

Where $E_q^x = \sum_{n=0}^{\infty}\binom{n}{2} = \frac{x^n}{[n]_q!}$ is a q-analogue of exponential function.

The q-analogue of $\Gamma(x)$ displays the following properties.

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x) \tag{1}$$

$$\Gamma_q(1) = 1$$

$$\Gamma_q(n+1) = [n]_q\,! \qquad n = 0,1,2,3,\ldots.$$

Where $[x]_q = \frac{1-q^x}{1-q}$. See Njionou Sadjan (2017).

It is recognized in literature (Mansour and Shaban, 2008) that the function $\psi_q(x)$ is increasing for $x > 0$.

In 1948, Wendel proved the inequalities

$$\left(\frac{x}{x+s}\right)^{1-s} \le \frac{\Gamma(x+s)}{x^s\Gamma(x)} \le 1 \tag{2}$$

For $x > 0$ and $s \in (0,1)$.

He used this to determine the asymptotic behavior of a ratio of Gamma functions.

In 1959, Gautschi proved the inequalities

$$x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq e^{\psi(x+1)(1-s)} \quad (3)$$

$$x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+1)^{1-s} \quad (4)$$

for $x > 0$ and $s \in (0,1)$.

In 2013, Laforgia and Natalini by using the classical mean value theorem extended

and refined inequality (3) as follows

$$e^{\psi(x+s)(1-s)} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq e^{\psi(x+1)(1-s)} \quad (5)$$

Inequalities that indicate bounds for the ratio of $\Gamma(x)/\Gamma(y)$, where x and y are some special form of numbers, have been vigorously studied by many researchers across the world. A comprehensive report on inequalities of this nature can be located in the survey article by Qi (2009) and the references therein.

The principal objective of this short article is to establish q-analogue of some known inequalities and some Inequalities bounding the ratio $\Gamma_q(x)/\Gamma_q(y)$, where $\Gamma_q(.)$ is the q-analogue of the Gamma function.

RESULTS and DISCUSSION

We begin with the following basic definition.

A function $f: (a, b) \to R^+$ is said to be logarithmically convex if the inequality

$$\log f(\alpha u + \beta v) \leq \alpha \log f(u) + \beta \log f(v)$$

Or equivalently

$$f(\alpha u + \beta v) \leq \left(f(u)\right)^{\alpha}\left(f(v)\right)^{\beta}$$

Holds for all $u, v \in (a, b)$ and α, β > 0 such that α + β = 1. (Merkle, 2008).

Theorem 2.1

Assume that $s \in (0,1)$ and $q \in (0,1)$. Then for any $x > 0$ the inequality

$$\left(\frac{[x]_q}{[x+s]_q}\right)^{1-s} \leq \frac{\Gamma_q(x+s)}{[x]_q{}^s\,\Gamma_q(x)} \leq 1 \quad (6)$$

is valid.

Proof. Recall that the function $\Gamma_q(x)$ is logarithmically convex. See Hashimu *et al*. (2022).

Hence,

$$\Gamma_q(\alpha u + \beta v) \leq \left(\Gamma_q(u)\right)^{\alpha}\left(\Gamma_q(v)\right)^{\beta} \quad (7)$$

for $u, v > 0$ and *α, β* > 0 such that *α* + *β* = 1.

Use the inequality (7) with $u = x$, $v = x + 1$, $\alpha = 1 - s$ and $\beta = s$. Also use it with $u = x + s$, $v = x + s + 1$, $\alpha = s$ and $\beta = 1 - s$.

The subsequent inequalities are;

$$\Gamma_q(x+s) < \left(\Gamma_q(x)\right)^{1-s}(\Gamma_q(x+1))^s = [x]_q{}^{s-1}\,\Gamma_q(x+1) \quad (8)$$

$$\Gamma_q(x+1) < \left(\Gamma_q(x+s)\right)^{s}(\Gamma_q(x+s+1))^{1-s} = [x+s]_q{}^{1-s}\;\Gamma_q(x+s) \quad (9)$$

Substituting (1) into (8) we obtain

$$\Gamma_q(x+s) < \left(\Gamma_q(x)\right)^{1-s}[x]_q{}^s\left(\Gamma_q(x)\right)^{s}$$

Which means

$$\Gamma_q(x+s) < [x]_q{}^s\;\Gamma_q(x) \quad (10)$$

Now, linking inequalities (9) and (10), we obtain

$$\frac{\Gamma_q(x+1)}{[x+s]_q{}^{1-s}} < \Gamma_q(x+s) < [x]_q{}^s\;\Gamma_q(x)$$

Which can be written as

$$\frac{[x]_q}{[x+s]_q{}^{1-s}}\Gamma_q(x) < \Gamma_q(x+s) < [x]_q{}^s\;\Gamma_q(x) \quad (11)$$

Finally, (11) can be rearranged as;

$$\left(\frac{[x]_q}{[x+s]_q}\right)^{1-s} \leq \frac{\Gamma_q(x+s)}{[x]_q{}^s\,\Gamma_q(x)} \leq 1$$

Which completes the proof of theorem 1.

Remark 2.1. Inequality (6) was demonstrated by Nantomah (2015) using a different technique. It provides a generalization of (2).

Theorem 2.2

Assume that $q \in (0,1)$ and $s \in (0,1)$. Then for any $x > 0$ the following inequality is valid.

$$[x]_q{}^{1-s} \leq \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \leq [x+]_q{}^{1-s} \quad (12)$$

Proof. Rearranging the inequality (8) gives the lower bound of theorem (2.2) as follows;

$$[x]_q{}^{1-s} \leq \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \quad (13)$$

While rearranging the inequality (9) and applying the trivial estimate $x + s < x + 1$ gives the upper bound of theorem (2.2) as follows;

$$\frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \leq [x+]_q{}^{1-s} \quad (14)$$

Lastly, combining inequalities (13) and (14), we obtain

$$[x]_q{}^{1-s} \leq \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \leq [x+]_q{}^{1-s}$$

Which concludes the proof of theorem 2.2

Remark 2.2 Inequality (12) is the q-analogue of (4).

Theorem 2.3

Let q ∈ (0, 1). Then the inequality

$$e^{(x-y)\psi_q(y)} < \frac{\Gamma_q(x)}{\Gamma_q(y)} < e^{(x-y)\psi_q(x)} \quad (15)$$

holds for $x > y > 0$.

Proof. Let H be defined for q ∈ (0,1) and t > 0 by $H(t) = \ln \Gamma_q(t)$. Further let (y, x) be fixed. Then by the classical mean value theorem, there exist a $\lambda \in (y, x)$ such that

$$H'(\lambda) = \frac{\Gamma_q(x) - \Gamma_q(y)}{x - y} = \psi_q(\lambda)$$

Recall that $\psi_q(t)$ is increasing for t > 0. See Mansour and Shabani (2008).

Then for $\lambda \in (y, x)$ we have

$$\psi_q(y) < \frac{1}{x-y} \ln \frac{\Gamma_q(x)}{\Gamma_q(y)} < \psi_q(y)$$

That is

$$(x-y)\psi_q(y) < \ln \frac{\Gamma_q(x)}{\Gamma_q(y)} < \psi_q(y)(x-y)$$

Subsequently, by taking exponents, we obtain the result (15).

Theorem 2.4

Let $q \in (0,1)$ and $s \geq 0$. Then the inequality

$$e^{\psi_q(x+s)(1-s)} \leq \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \leq e^{\psi_q(x+1)(1-s)} \quad (16)$$

holds for $x > 0$.

Proof. This follows from Theorem (2.3) upon replacing x and y respectively by $x + 1$ and $x + s$.

Theorem 2.5

Let $q \in (0,1)$. Then the inequality

$$e^{\frac{1}{2}\psi_q(x+\frac{1}{2})} \leq \frac{\Gamma_q(x+1)}{\Gamma_q\left(x+\frac{1}{2}\right)} \leq e^{\frac{1}{2}\psi_q(x+1)} \quad (17)$$

Holds for $x > 0$.

Proof. This follows right from theorem (2.4) by letting $s = \frac{1}{2}$.

Remark 2.3. Inequality (16) is the q-analogue of (5) which was established by Laforgia and Natalini in 2013.

Remark 2.4. The inequality (15) provides a q-analogue of the result'

$$e^{(x-y)\psi(y)} < \frac{\Gamma(x)}{\Gamma(y)} < e^{(x-y)\psi(x)} \quad (18)$$

holds for $x > y > 0$, which was established in (Qi, 1999) [corollary 2].

Conclusion

We have established q- analogues of some known inequalities for functions involving the classical Gamma function and some inequalities bounding the ratio $\Gamma_q(x)/\Gamma_q(y)$, where $\Gamma_q(.)$ is the q-analogue of the Gamma function. The proven result provided the q-analogue of some known result concerning the classical Gamma function.

CONFLICT OF INTEREST

The author declares no conflict of interest.

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